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A[1] WALKER'S GUIDE TO DECOHERENCE TIMES IN STOCHASTIC COOLING

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I. Introduction

Since there seems to be some confusion about the statistical processes inherent in stochastic cooling I give derivations of two probability density distributions that may serve to illuminate the situation.

In the ideal case the statistical fluctuation sensed survives to the correction station essentially intact but is smeared out to present a statistically independent sample by the time of the next pass through the sensing station. Any physically realizable system must necessarily be a compromise of these ideal conditions.

I consider two problems: 1) What is the time dependent probability density distribution of the number of particles in a sample of beam, and 2) What is the time dependent probability density distribution of the difference of the number of particles on either side of a sample of beam; with given initial conditions?

II. Unsplit Sample

Consider a sample of a beam containing, on the average, $\langle n \rangle$[2] particles. Let λ be the probability per unit time that any one particle leaves the sample (due e.g. to momentum spread). We seek the probability, $P_m(t)$, that m particles are present at time, t, if there are exactly m_o present at time zero. From elementary considerations,

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[1] Random.

[2] The notation $\langle\ \rangle$ signifies expectation value.

$$P_m(t + \delta t) = P_{m-1}(t)\ \lambda\ \langle n\rangle\ \delta t + P_{m+1}(t)\ \lambda(m+1)\ \delta t + P_m(t)(1-\lambda(\langle n\rangle+m)\delta t) \quad ,$$

and letting $\delta t \rightarrow 0$,

$$\frac{dP_m}{dt} = P_{m-1}\ \lambda\ \langle n\rangle + P_{m+1}\ \lambda(m+1) - P_m\ \lambda\ (\langle n\rangle + m) \tag{1}$$

This can be solved exactly be introducing the generating function

$$\varphi = \sum_m P_m\ x^m$$

and multiplying Eq. 1 by x^m and summing we get

$$\frac{\partial\varphi}{\partial t} = (1 - x)\left\{-\lambda\langle n\rangle\ \varphi + \lambda\ \frac{\partial\varphi}{\partial x}\right\} \quad .$$

Since $\langle n\rangle$, m_o, $m \gg 1$ the Central Limit Theorem of statistics says that P_m will be very well approximated by the normal distribution (with mean and variance a function of time) so it is sufficient for our purposes to determine only the mean ($\langle m\rangle$) and variance ($\langle m^2\rangle - \langle m\rangle^2$) of m.

Multiplying Eq. 1 by m and summing over m,

$$\frac{d\langle m\rangle}{dt} = \lambda\ (\langle n\rangle - \langle m\rangle)$$

or

$$\langle m\rangle = \langle n\rangle - e^{-\lambda t}\ (\langle n\rangle - m_o) \quad .$$

Multiplying Eq. 1 by m^2 and summing over m,

$$\frac{d\langle m^2\rangle}{dt} = \lambda\ (\langle n\rangle + \langle m\rangle\ (2\langle n\rangle + 1) - 2\langle m^2\rangle) \quad ,$$

whose solution is

$$\langle m^2\rangle = \langle m\rangle + \langle m\rangle^2 - m_o\ e^{-2\lambda t} \quad ,$$

or

$$\mathrm{var}\ (m) = \langle m \rangle - m_o\ e^{-2\lambda t} = \sigma_m^2 \ .$$

The time dependent probability density distribution of m is

$$P_m(t) = (2\pi\ \sigma_m^2)^{-\frac{1}{2}}\ \exp\left\{-\frac{(m - \langle m \rangle)^2}{2\sigma_m^2}\right\} \ .$$

III. Split Sample

Consider a sample of beam containing N particles. If there are m and n particles (initially m_o and n_o) on either side of the beam the sensed transverse beam position is

$$\bar{y} = \frac{m-n}{m+n}\,\sigma_{beam} = \frac{m-n}{N}\,\sigma_{beam} \ .$$

Let υ equal the probability per unit time (due e.g. to betatron oscillations) that a particle will change sides. The joint probability distribution $P_{m,n}(t)$ satisfies

$$\frac{dP_{m,n}}{dt} = P_{m-1,\,n+1}\,\upsilon\,(n+1) + P_{m+1,\,n-1}\,\upsilon\,(m+1) - P_{m,n}\,\upsilon\,(m+n) \ . \qquad (2)$$

As in II, we calculate only the mean and variance of m-n. Since m and n are not statistically independent we have

$$\mathrm{var}(m-n) = \mathrm{var}(m) + \mathrm{var}(n) - 2\ \mathrm{cov}(m,n)$$

or

$$\mathrm{var}(m-n) = \langle m^2 \rangle + \langle n^2 \rangle - (\langle m \rangle - \langle n \rangle)^2 - 2\langle mn \rangle \ .$$

Multiplying Eq. 2 by n, m, n^2, m^2, and mn and each time summing over m, n, we get

$$\langle m - n \rangle = (m_o - n_o)\ e^{-2\upsilon t} \ ,$$

and, after a moderate amount of algebra

$$\mathrm{var}(m-n) = N(1 - e^{-4\upsilon t}) = \sigma_{m-n}^2 \ .$$

The time dependent probability density distribution of m-n is

$$P(t)_{m-n} = (2\pi\,\sigma^2_{m-n})^{-\frac{1}{2}} \exp\left\{ - \frac{[(m-n)-\langle m-n\rangle]^2}{2\sigma^2_{m-n}} \right\} .$$

IV. Conclusion

Using the above, and the assumption that the initial (t=0) conditions are normally distributed, the interested reader, with a little arithmetic (or computer time), can calculate the odds[3] on the success of a particular stochastic cooling system.

3) "For most men ('till by losing rendered sager) will back their own opinions with a wager"

George Gordon, Lord Byron.

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