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Accelerator Division
Technical Note

AGS/AD/ Tech. Note No. 400

LONGITUDINAL MODE COUPLING

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August 22, 1994

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ABSTRACT

In this article, we present a formulation for the longitudinal mode coupling, which is shown to be an eigenvalue problem. Several important properties of the longitudinal mode coupling are discussed. A special case of the coupling between the azimuthal mode $m = \pm 1$ is discussed in comparison with the Robinson approach for the RF beam loading problem. Finally, the relation between the presented approach and the dispersion type approach for the mode coupling will be discussed.

* Work performed under the auspices of the U.S. Department of Energy

I. Introduction

In the longitudinal bunched beam motion, if azimuthal modes are coupled by the feedback through the impedances of the surroundings environment, then an instability will occur. This coupling may happen when the beam current reaches a threshold, then a large negative imaginary part of eigenvalue will abruptly build up. Once a mode coupling happens, usually the instability mechanism will be very strong [1-7].

The longitudinal mode coupling is the most fundamental one among the various kinds of mode couplings. In this article, a formulation of the longitudinal mode coupling is presented, which is shown to be an eigenvalue problem. The interaction matrix of the mode coupling is decomposed by the impedance and Hankel spectrum matrices. Several important properties of the mode coupling are discussed.

One of the most important cases in the longitudinal mode coupling is the coupling between the azimuthal mode $m = \pm 1$, which is discussed in comparison with the conventional beam loading treatment using the Robinson approach. The upgraded AGS RF beam loading is used as an example to show the similarities and the differences of the two approaches.

Yet another approach in solving the mode coupling is using a dispersion type equation [3,7]. The relation between these two approaches will be discussed in the last section.

II. Sacherer Integral Equation

Consider the Sacherer integral equation [8],

$$(\omega - m\,\omega_S)R^{(m)}(r) = j^{m+1} m\,\omega_S \xi W(r) \sum_{p=-\infty}^{\infty} \frac{Z(p)}{p} J_m(pr) \sum_{m'=-\infty}^{\infty} j^{-m'} \Lambda^{(m')}(p) \qquad (2\text{-}1)$$

where ω is the coherent frequency shift, m is the azimuthal mode number, ω_S is the synchrotron frequency, and $R^{(m)}(r)$ is the radial function. The scaling factor ξ is defined as,

$$\xi = \frac{2\pi I_0}{V\cos\phi_S} \qquad (2\text{-}2)$$

where I_0 is the average beam current, V is the total RF voltage and ϕ_S is the stable phase. Note that ξ represents the influence of the beam intensity to the beam instability. The weight function is

$$W(r) = -\frac{d\psi_0}{dr}\frac{1}{r} \tag{2-3}$$

where ψ_0 is the stationary particle distribution. The Hankel spectrum of the radial function is defined as,

$$\Lambda^{(m)}(p) = \int_0^\infty R^{(m)}(r) J_m(pr) r dr \tag{2-4}$$

and finally, $Z(p)$ is the impedance sampled at the pth harmonic.

For the weight function $W(r)$, a set of normalized orthogonal polynomials $f_k^{(m)}(r)$ can always be found such that,

$$\int_0^\infty W(r) f_k^{(m)}(r) f_l^{(m)}(r) r dr = \delta_{k,l} \tag{2-5}$$

Using the orthogonal polynomial, the radial function can be written as,

$$R^{(m)}(r) = W(r) \sum_{k'=0}^{\infty} \alpha_{k'}^{(m)} f_{k'}^{(m)}(r) \tag{2-6}$$

where $\alpha_k^{(m)}$ is the coefficient to be solved.

The Hankel spectra for these orthogonal polynomials can be written as,

$$\Lambda_k^{(m)}(p) = \int_0^\infty W(r) f_k^{(m)}(r) J_m(pr) r dr \tag{2-7}$$

The relation between the two Hankel spectra of (2-4) and (2-7) is,

$$\Lambda^{(m)}(p) = \sum_{k'=0}^{\infty} \alpha_{k'}^{(m)} \Lambda_{k'}^{(m)}(p) \tag{2-8}$$

The Bessel function $J_m(pr)$ can be expanded by the orthogonal polynomials and the associated Hankel spectra,

$$J_m(pr) = \sum_{k'=0}^{\infty} \Lambda_{k'}^{(m)}(p) f_{k'}^{(m)}(r) \tag{2-9}$$

Substituting (2-6), (2-8) and (2-9) into (2-1), we have,

$$(\omega - m\,\omega_S)W(r)\sum_{k'=0}^{\infty}\alpha_{k'}^{(m)}f_{k'}^{(m)}(r) = j^{m+1}m\,\omega_S\,\xi W(r)\sum_{p=-\infty}^{\infty}\frac{Z(p)}{p}$$
$$\times\sum_{k',l'=0}^{\infty}\Lambda_{k'}^{(m)}(p)f_{k'}^{(m)}(r)\sum_{m'=-\infty}^{\infty}j^{-m'}\alpha_{l'}^{(m')}\Lambda_{l'}^{(m')}(p) \tag{2-10}$$

We multiply (2-10) by $f_k^{(m)}(r)r$ and integrate over r, also define the interaction submatrix of the azimuthal modes of m and m',

$$\mathrm{M}^{(m,m')} = j^{m+1-m'}m\,\omega_S\,\xi$$

$$\times\begin{bmatrix} \sum_{p=-\infty}^{\infty}\frac{Z(p)}{p}\Lambda_0^{(m)}(p)\Lambda_0^{(m')}(p) & \cdots & \sum_{p=-\infty}^{\infty}\frac{Z(p)}{p}\Lambda_0^{(m)}(p)\Lambda_{\bar{k}}^{(m')}(p) \\ \vdots & \vdots & \vdots \\ \sum_{p=-\infty}^{\infty}\frac{Z(p)}{p}\Lambda_{\bar{k}}^{(m)}(p)\Lambda_0^{(m')}(p) & \cdots & \sum_{p=-\infty}^{\infty}\frac{Z(p)}{p}\Lambda_{\bar{k}}^{(m)}(p)\Lambda_{\bar{k}}^{(m')}(p) \end{bmatrix} \tag{2-11}$$

Now we are ready to give the Sacherer integral equation with mode coupling in a form of algebraic equations. To be not overwhelmed by large dimensions, firstly we consider the coupling only between m and m',

$$\begin{bmatrix} (\omega - m\,\omega_S)\mathrm{I}_{\bar{k}+1} & 0 \\ 0 & (\omega - m'\,\omega_S)\mathrm{I}_{\bar{k}+1} \end{bmatrix}\begin{bmatrix} \alpha^{(m)} \\ \alpha^{(m')} \end{bmatrix} = \begin{bmatrix} \mathrm{M}^{(m,m)} & \mathrm{M}^{(m,m')} \\ \mathrm{M}^{(m',m)} & \mathrm{M}^{(m',m')} \end{bmatrix}\begin{bmatrix} \alpha^{(m)} \\ \alpha^{(m')} \end{bmatrix} \tag{2-12}$$

where $\mathrm{I}_{\bar{k}+1}$ is an identity matrix with dimension $\bar{k}+1$, and

$$\alpha^{(m)} = [\,\alpha_0^{(m)}\ldots\alpha_{\bar{k}}^{(m)}\,]^T \tag{2-13}$$

is the eigenvector of the mth azimuthal mode, where the superscript T denotes transpose. The equation (2-12) can be easily extended to include all necessary modes.

III. Solution of Mode Coupling

We rewrite the equation (2-12) as,

$$\omega\begin{bmatrix} \alpha^{(m)} \\ \alpha^{(m')} \end{bmatrix} = \left[\begin{bmatrix} m\,\omega_S\mathrm{I}_{\bar{k}+1} & 0 \\ 0 & m'\,\omega_S\mathrm{I}_{\bar{k}+1} \end{bmatrix} + \begin{bmatrix} \mathrm{M}^{(m,m)} & \mathrm{M}^{(m,m')} \\ \mathrm{M}^{(m',m)} & \mathrm{M}^{(m',m')} \end{bmatrix}\right]\begin{bmatrix} \alpha^{(m)} \\ \alpha^{(m')} \end{bmatrix}$$

$$= [\ \mathbf{I}^{(m,m')} + \mathbf{M}\ ] \begin{bmatrix} \alpha^{(m)} \\ \alpha^{(m')} \end{bmatrix} \tag{3-1}$$

then to solve the mode coupling we need only to find the eigenvalues and the eigenvectors of the system matrix $\mathbf{I}^{(m,m')} + \mathbf{M}$, which is discussed as follows.

1. Note that the interaction matrix $\mathbf{M}$ is zero if the beam current $I_0 = 0$. The eigenvalues then simply become,

$$\omega_{(i)} = m\,\omega_S, \qquad i = 0, \cdots, \bar{k} \tag{3-2}$$

with the corresponding eigenvectors

$$\alpha_{(i)} = [\ 0, \cdots, 0, \alpha_i^{(m)}, 0, \cdots, 0\ ]^T, \qquad \alpha_i^{(m)} = 1 \tag{3-3}$$

and

$$\omega_{(i)} = m'\,\omega_S, \qquad i = \bar{k}+1, \cdots, 2\bar{k}+1 \tag{3-4}$$

with the corresponding eigenvectors

$$\alpha_{(i)} = [\ 0, \cdots, 0, \alpha_i^{(m')}, 0, \cdots, 0\ ]^T, \qquad \alpha_i^{(m')} = 1 \tag{3-5}$$

If the beam current is not zero, but the interaction submatrices between different azimuthal modes are zero, i.e. $\mathbf{M}^{(m,m')} = \mathbf{M}^{(m',m)} = 0$, then the system is decoupled into two independent systems with different azimuthal modes. In specific, there are total $2\bar{k}+2$ eigenvalues. For $\omega_{(i)}$, $i = 0, \cdots, \bar{k}$, the eigenvectors are $\alpha_{(i)} = [\ \alpha^{(m)}\ \ 0\ ]^T$, and for $\omega_{(i)}$, $i = \bar{k}+1, \cdots, 2\bar{k}+1$, the eigenvectors are $\alpha_{(i)} = [\ 0\ \ \alpha^{(m')}\ ]^T$. Thus, one may instead just study the following subsystem,

$$(\omega - m\,\omega_S)\,\alpha^{(m)} = \mathbf{M}^{(m,m)}\,\alpha^{(m)} \tag{3-6}$$

which is an eigenvalue problem. Since the scaling of the eigenvector $\alpha^{(m)}$ is arbitrary, therefore from (2-6), the scaling of radial modes is arbitrary. Note that the eigenvalues are proportional to the beam intensity, meanwhile the radial modes are invariant with respect to the beam intensity. For an inductive impedance, the matrix $\mathbf{M}^{(m,m)}$ is real and symmetric, therefore all eigenvalues are real, which implies that there is no mechanism of instability. In a non-coupled system, all eigenvalues are proportional to the beam intensity, therefore there

is no mode coupling within an azimuthal mode.

2. Consider the general case of mode coupling (3-1), where each element in the eigenvector $[\,\alpha^{(m)} \quad \alpha^{(m')}\,]^T$ is the coefficient of the corresponding polynomial of the radial mode in expansion, as shown in (2-6). It is known that if the orthogonality between the polynomials in expansion is available, then these polynomials constitute an orthogonal base. A set of eigenvalues and eigenvectors is a solution for the equation, which has an explicit physical interpretation based on the orthogonal base. This is the situation of the azimuthally non-coupled system discussed previously, such as that of (3-6). In general, if the orthogonality of the polynomials is not available, then there exists redundancy in the system, which usually implies complications in solving the problem. In the mode coupling problem of (3-1), in general the orthogonality between the expansion polynomials of the different azimuthal modes are not available. This fact raises a question of the eligibility of the equation (3-1) to be treated as an eigenvalue problem. We note that the orthogonality of the polynomials for the same m is guaranteed, and between different azimuthal modes the radial functions along with the rotation factor $e^{jm\theta}$ represent particle distribution in phase space as,

$$\psi_p(r,\theta) = \sum_{m'=-\infty}^{\infty} R^{(m')}(r) e^{jm'\theta} \tag{3-7}$$

therefore the orthogonality between the polynomials in different azimuthal modes is implicitly implied by the orthogonality of the rotation factors. The conclusion is that the equation of the mode coupling (3-1) is eligible to be treated as an eigenvalue problem.

3. In the general case, because of the interaction between the different azimuthal modes, the eigenvalue ω is no longer linearly proportional to the beam intensity, and also the radial modes will not be invariant with the beam intensity. In one of the extreme cases where $I_0 = 0$, the radial modes are directly associated with the corresponding orthogonal polynomials, for instance the ith radial mode is $R_{(i)}^{(m)}(r) = W(r) f_i^{(m)}(r)$. Another extreme case is that the norm of $\mathbf{M}$ is much larger than the one of $\mathbf{I}^{(m,m')}$, such that the radial mode can somewhat recover the invariance with respect to the beam intensity. The most interest-

ing case unfortunately is the one between these two extreme situations. In summary, for a general mode coupling problem the eigenvalues vary along, but not linearly, with the beam intensity, and also the radial modes keep changing when the beam current increases.

4. To solve the problem, the eigenvalues and the corresponding eigenvectors of (3-1) can be found for different beam intensities. For each eigenvalue, the eigenvector can be used to find the corresponding radial mode within an azimuthal mode, as in (2-6), then by combining the different azimuthal modes, the perturbative particle distribution can be found, as in (3-7).

IV. Further Study of Mode Coupling

To have a further study of the longitudinal mode coupling, several interesting properties of the interaction matrix need to be discussed. To do so, we define an impedance matrix

$$\mathbf{Z} = diag\{\frac{Z(p)}{p}\} = \begin{bmatrix} \frac{Z(-\bar{p})}{-\bar{p}} & 0 & \cdots & 0 \\ 0 & \frac{Z(-\bar{p}+1)}{-\bar{p}+1} & . & 0 \\ . & . & . & . \\ 0 & 0 & \cdots & \frac{Z(\bar{p})}{\bar{p}} \end{bmatrix} \tag{4-1}$$

and a Hankel spectrum matrix

$$\Lambda^{(m)} = \begin{bmatrix} \Lambda_0^{(m)}(-\bar{p}) & \cdots & \Lambda_0^{(m)}(\bar{p}) \\ . & . & . \\ . & . & . \\ \Lambda_{\bar{k}}^{(m)}(-\bar{p}) & \cdots & \Lambda_{\bar{k}}^{(m)}(\bar{p}) \end{bmatrix} \tag{4-2}$$

Then the interaction matrix can be decomposed as,

$$\mathbf{M}^{(m,m')} = j^{m+1-m'} m\, \omega_S \xi\, \Lambda^{(m)}\, \mathbf{Z}\, \Lambda^{(m')^T} \tag{4-3}$$

From (4-3) it is clear that the mode coupling will be affected by the Hankel spectrum $\Lambda^{(m)}$, the associated impedance $\mathbf{Z}$, and the Hankel spectrum $\Lambda^{(m')}$. The Hankel spectrum of the first five orthogonal polynomials for the azimuthal modes 1 to 4 are shown in Fig.1, where a Gaussian distribution with an effective half bunch length of 1 *rad* is used. It can

be observed that like the Bessel function $J_m(pr)$, if m is odd, the Hankel spectrum $\Lambda_k^{(m)}(p)$ is odd with respect to the frequency, if m is even, the Hankel spectrum is also even with respect to the frequency.

Consider a resonator impedance $Z = Z_R + jZ_I$. The real part of the impedance $Z_R(p)/p$ is odd with respect to the frequency, and the imaginary part $Z_I(p)/p$ is even. A resonator impedance $Z(p)/p$ is plotted in Fig.2, where the resonator is tuned at $p = 5$, i.e. the resonant frequency is 5 times of the RF frequency. The shunt resistance used is 4.2 $K\Omega$ and the quality factor is 3.

Under the condition that the impedance $Z(p)$ is picked at $p\,\omega_{RF}$, we have the following observation.

If the interaction index $m-m'$ is even, then $\Lambda^{(m)}\Lambda^{(m')^T}$ is even, the imaginary part of the impedance Z_I is left on and the real part vanishes. Since $j^{m+1-m'}$ is imaginary, $\mathrm{M}^{(m,m')}$ is real.

If $m-m'$ is odd, then $\Lambda^{(m)}\Lambda^{(m')^T}$ is odd, the real part of the impedance Z_R is left on and the imaginary part vanishes. Since $j^{m+1-m'}$ is real, $\mathrm{M}^{(m,m')}$ is also real.

We may conclude that the matrix $\mathbf{M}$ in (3-1) is always a real matrix, whose components may be however from either the real or imaginary part of the impedance according to the interaction index $m-m'$. This is the fundamental mechanism of the mode coupling.

Consider a simple example for a mode coupling between m and m', assuming that both modes are positive and m is smaller than m'. One needs to find the eigenvalues of the system matrix, i.e. to solve the following equation,

$$\left| \begin{bmatrix} \omega - m\,\omega_S & 0 \\ 0 & \omega - m'\,\omega_S \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = 0 \tag{4-4}$$

where a,b,c,d are real variables of the beam intensity. The solutions of the (4-4) are

$$\omega = \frac{1}{2}\left\{ (m'+m)\omega_S - a - d \pm \left[\left((m'-m)\omega_S + a - d \right)^2 + 4bc \right]^{1/2} \right\} \tag{4-5}$$

If the beam intensity vanishes, the variable $A = ((m'-m)\omega_S + a - d)^2 + 4bc$ is positive. When the beam intensity increases, if the variable A also increases then the two modes repel each other, and otherwise the modes attract each other. In the later case shortly before the variable A turns from positive to negative the real parts of the solutions approaches to each other abruptly, and shortly after that point the imaginary parts of the solutions depart from zero abruptly, and become a positive one and a negative one. The negative imaginary part implies the instability. A typical mode coupling is shown in Fig.3. Note that however not every merge of the modes is mode coupling, a simple mode crossing is shown in Fig.4, where no mode coupling mechanism exists.

The real situation of the mode coupling is however much more complicated, it is discussed from the following aspects.

1. In a mode coupling problem, usually more than only two modes are involved. In fact, every influential mode which interacts with others should be considered. To identify these modes, the equation (4-3) can be used, which indicates that only if the influence of either $\mathbf{A}^{(m)}$, $\mathbf{Z}$, or $\mathbf{A}^{(m')}$ for a particular mode vanishes, then the interaction submatrix $\mathbf{M}^{(m,m')}$ can be eliminated. Should some modes be improperly eliminated, then the solution of the coupling would be significantly distorted. An example is shown in Fig.5, where in Fig.5a, a coupling between the azimuthal modes from -4 to 4 is considered for a beam whose Hankel spectrum can be represented by that of Fig.1. The impedance is the RF cavity tuned at $p = 1$, and therefore whose influence outside the modes from -4 to 4 can be disregarded. In Fig.5b, using the same parameters of the beam and the impedance only the coupling between the modes -1 and 1 is calculated. Apparently, the mode coupling shown in Fig.5a is more reliable and the simplification used in Fig.5b, which disregards all higher azimuthal modes, is not acceptable.

2. By studying the interaction matrix, we get the following useful relations between the interaction submatrices.

$$\mathbf{M}^{(m,m')}/m = (-1)^{m-m'}\mathbf{M}^{(m',m)^T}/m' \tag{4-6}$$

and

$$\mathbf{M}^{(m,m')} = \mathbf{M}^{(m,-m')} \tag{4-7}$$

Applying (4-6) and (4-7), we also get

$$\mathbf{M}^{(m,m')} = -\mathbf{M}^{(-m,m')} \tag{4-8}$$

To estimate the strength of the mode coupling, a useful property in the eigenvalue problem is that if the system matrix is real and symmetrical then all eigenvalues are real, i.e. there exists no imaginary part of the solution, and therefore exists no mode coupling. We note that if $Z(p)$ is sampled at $p\,\omega_{RF}$ then the system matrix $\mathbf{I}^{(m,m')} + \mathbf{M}$ is real. For an even interaction index $m-m'$, if the scalings of m and m' are disregarded, then the interaction matrix is symmetrical as shown in (4-6), which suggests that the modes will not attract each other. On the other hand, for an odd $m-m'$ the interaction matrix is antisymmetrical, and this offers a possibility of mode coupling. The second factor which affects the coupling strength can be observed from the Hankel spectra of different azimuthal modes shown in Fig.1. Since the coupling strength is determined by $\Lambda^{(m)}\Lambda^{(m')^T}$ for a given impedance, it is obvious that a mode interacts more strongly with the adjacent modes than the other modes. Upon these results, we may conclude that among all interactions for the azimuthal mode m, the ones with the modes $m \pm 1$ will be the strongest, and also there is good chance for mode couplings to happen.

An interesting example is that if the impedance is inductive, then no real part of the impedance exists. Because all interaction submatrices between the mth mode and the adjacent modes are zero, therefore there exists no mode coupling.

A special case is the coupling between the azimuthal modes -1 and 1, where because of the sign change of the element m in the interaction submatrix $\mathbf{M}^{(m,m')}$, as shown in (4-3), the situation is somewhat different from the ones discussed above. This will be discussed in Section V.

3. Sometimes, one may use only the first orthogonal polynomial for each azimuthal mode to represent the radial mode. This may be allowed under certain conditions. An example is shown in Fig.6, where the bunch and impedance parameters are the same as that used in Fig.5, except that in the calculation of Fig.5 the expansion of 4 orthogonal polynomials for a radial mode is used, whereas in Fig.6 only the first polynomial is used. Comparing the Fig.6a with Fig.5a, and Fig.6b with Fig.5b, respectively, one may find that the main features of the mode coupling remain after the model simplification. It is reminded that in both calculations, the narrowband resonator is tuned to $p = 1$. By examining the Hankel spectra shown in Fig.1, it is not difficult to find that with this impedance, the first orthogonal polynomials are the dominant ones comparing with other polynomials. This fact offers an opportunity for the model simplification. By simply tuning the same resonator to $p = 5$, the situation is changed, which is shown in Fig.7. The simplified model with only the first orthogonal polynomials in Fig.7b has missed all couplings shown in Fig.7a, and therefore the simplification becomes not acceptable.

4. In fact, because $Z(p)$ should be picked up at $p\,\omega_{RF} + m\,\omega_S$ rather than at $p\,\omega_{RF}$, a small imaginary portion will be left on in the interaction matrix $\mathbf{M}$, which in some cases becomes important, as shown in the following section.

V. Coupling Between Azimuthal Modes 1 and -1

In the last section, it is shown that the coupling of a mode with adjacent azimuthal modes in general will dominate the couplings comparing with others. Yet the coupling between 1 and -1 is an important case in the longitudinal mode coupling, and it is somewhat different from the others.

Using

$$\Lambda^{(-1)} = -\,\Lambda^{(1)} \tag{5-1}$$

we may write the interaction matrix as

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}^{(1,1)} & \mathbf{M}^{(1,-1)} \\ \mathbf{M}^{(-1,1)} & \mathbf{M}^{(-1,-1)} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_1 & \mathbf{M}_1 \\ -\mathbf{M}_1 & -\mathbf{M}_1 \end{bmatrix} \tag{5-2}$$

where

$$\mathbf{M}_1 = j\omega_S \xi \, \mathbf{\Lambda}^{(1)} \, \mathbf{Z} \, \mathbf{\Lambda}^{(1)^T} \tag{5-3}$$

We note that because of the sign changes from $m = 1$ to $m = -1$, although the interaction index is even, the interaction matrix is antisymmetrical, as shown in (5-2), which provides a possibility of a mode coupling.

To simplify the discussion, we assume that only the first orthogonal polynomial is used in expansion, then the matrix $\mathbf{M}_1$ becomes a scale, written as M_1. Thus, the eigenvalues can be found from,

$$| \, \omega \mathbf{I} - \mathbf{I}^{(1,-1)} - \mathbf{M} \, | = \omega^2 - 2\omega_S M_1 - \omega_S^2 = 0 \tag{5-4}$$

As far as the mode coupling is concerned, we only have to consider the real part of M_1, written as M_R, which is from Z_I. From (5-4), it is found that if $M_R > 0$, then the two modes expel each other, and if $M_R < 0$, they attract each other. In the later case, if $M_R = -\omega_S/2$, then $\omega = 0$. If the beam intensity further increases, then a negative imaginary part of the eigenvalue is generated, which is unstable. This is a typical mode coupling, which will be shown later to be equivalent to the second Robinson instability.

We also note that a small imaginary part of M_1 exists, which is from the real part of the impedance Z_R. Considering a narrowband resonator such as an RF cavity, it is not difficult to find that the sign of both parts are determined by the cavity detuning. Using the Laplace operator $s = j\omega$, and assuming $\omega \approx \omega_S$, then the equation (5-4) becomes

$$s^2 + 2M_I s + \omega_S^2 = 0 \tag{5-5}$$

where M_1 in (5-4) is replaced by jM_I, with M_I being the imaginary part and whose sign is determined by the cavity detuning. The effect of the real part of the impedance, or the imaginary part of the interaction matrix, is equivalent to the Robinson damping, or the first Robinson instability.

In Fig.8a, a mode coupling between the azimuthal modes $m = \pm 1$ is plotted, where the loci of the eigenvalues are shown in plane of real and imaginary parts of eigenvalue. The lines with small dots are for the case that the RF cavity is detuned in correct direction, each dot represents a step of the beam current increment. The lines of bold dots are for the case that the cavity is detuned in the opposite direction. The Fig.8b is a blow-up of the same plot, where one observes that if the cavity detuned in a correct direction and the beam current increases, the two dominant eigenvalues move away from $m = \pm 1$ on the real axis, generating positive imaginary parts, which imply damping. After reaching a beam current threshold, a large negative imaginary part is generated, and a strong instability occurs. It is also observed that if the cavity is detuned in a wrong direction, a small antidamping happens immediately after the beam current moves away from zero. Also we note that around a beam current threshold, the modes of $m = 1$ and $m = 2$ start to strongly repel each other without a clear mode coupling, generating a strong instability.

It is indicated that the small imaginary parts of the eigenvalues generated before the mode couplings are from the small imaginary portion of the interaction matrix. By eliminating these imaginary elements in the interaction matrix, the problem becomes a 'pure' mode coupling, which is shown in Fig.9a, and a blow-up in Fig.9b, for the same parameters in plotting Fig.8.

We may conclude that for the mode coupling between the modes $m = \pm 1$, the weak damping or antidamping shown in Fig.8 are from the imaginary portion of the interaction matrix, which in this case is from the real part of the impedance of the RF cavity. This is the first Robinson instability criterion. On the other hand, the strong instability generated from the mode coupling is from the real portion of the interaction matrix, which in this case is from the imaginary part of the cavity impedance. This is associated with the second Robinson instability criterion.

It is interesting to compare the mode coupling between $m = \pm 1$ with the Robinson approach. In the Robinson approach, only the beam fundamental frequency is concerned.

The beam instability can be represented by a 4th order dynamical equation,

$$s^4 + 2\sigma s^3 + (\sigma^2 (1 + \tan^2\phi_Z) + \omega_S^2)s^2 + 2\sigma\omega^2 s$$
$$+ \sigma^2 (1 + \tan^2\phi_Z) \omega_S^2 - I_B R \omega_S^2 \sigma^2 \tan\phi_Z/(V\cos\phi_S) = 0 \qquad (5\text{-}6)$$

where σ is the half bandwidth of the cavity, ϕ_Z is the cavity detune angle, I_B is the beam fundamental current, R is the cavity shunt resistance.

Using the same parameters, the root loci are plotted in Fig.10a. We note that since the transfer function of the RF cavity is of second order, in addition to the fundamental modes of $m = \pm 1$, there exist other two roots as shown in the plot, which have large positive imaginary part, i.e. with a fast damping, and therefore are the minor roots. These roots are however the source of the small imaginary parts of the dominant roots, which are shown in the blow-up in Fig.10b. Without these minor roots, the problem also becomes a 'pure' mode coupling, in the sense that only two 'purely' fundamental modes are concerned.

By comparing Fig.10b with Fig.8b, the similarities and the differences of the mode coupling approach and the Robinson approach can be found. The following is a brief summary of the comparison. In the mode coupling approach, more than fundamental modes are considered, the impedances are considered with the real and imaginary parts, respectively, and the particle distribution is included in the calculation. In the Robinson approach, only the fundamental modes are concerned, the impedances are considered in a form of second order transfer function, and the particle distribution is only reflected in the ratio of the beam DC current and the fundamental current. For example, if the bunch is parabolic with half bunch length of 1.57 *rad*, then this ratio is taken as 1.6, and for short bunches it is taken as 2 [9].

In this example, one may notice that the results of the mode coupling between the modes $m = \pm 1$ obtained by the two approaches are not much different. This is however not always the case. Taking the upgraded AGS RF system at injection as an example, where the RF frequency is 4.2 *MHz* and the synchrotron frequency is 1.78 *KHz*. The impedance considered is the RF cavity, with a 4.2 $K\Omega$ shunt resistance, and a quality factor

20. To accommodate the beam loading with 6×10^{13} protons per ring, with the RF voltage 40 KV, the cavity is detuned up by 58.2 KHz. The half bandwidth of the cavity is therefore roughly equal to $\sigma = \omega_R/(2Q) \approx 6.6 \times 10^5$ rad/sec.

Since the cavity detuning angle is in correct direction, the first Robinson criterion for the beam stability is satisfied. Using (5-6), it is found that at $I_B = 22.8\ A$ the beam instability occurs. For a Gaussian distribution with a half bunch length of 1 rad, the ratio between the beam fundamental current I_B and the DC current I_0 is roughly 1.77, which shows that the corresponding I_0 for the instability is 12.9 A. Using the mode coupling between the modes $m = \pm 1$, the result is calculated and shown in Fig.5b, where each radial mode is expanded by 4 orthogonal polynomials. It can be observed that one of the radial modes will be coupled at the beam current around 16.2 A, which is slightly higher than the result obtained from the Robinson approach.

In fact, the coupling between the modes $m = \pm 1$ must be calculated by considering the influence of other azimuthal modes. The result is shown in Fig.5a, where the azimuthal modes from -4 to 4 are included. What we find is that because of the influence of other azimuthal modes, the coupling shown in Fig.5b will not happen at all. This is an example showing that the a complete model of the mode coupling is probably a more reliable means in determining the beam instability, even just for the simple case of coupling between $m = \pm 1$. Note that in this study the influence of the potential well distortion and the Landau damping is not included.

VI. Comparison with Dispersion Relation Type Solution

Another important approach to the mode coupling is a dispersion type solution [3,7]. Consider the Sacherer integral equation in the version of Hankel harmonic sampling,

$$\lambda^{(m)}\, j^{-m} \Lambda_p^{(m)} = \mathrm{K}^{(m)} \sum_{m'=-\infty}^{\infty} j^{-m'} \Lambda_p^{(m')} \tag{6-1}$$

where $\lambda^{(m)} = \omega - m\,\omega_S$, and

$$\Lambda_p^{(m)} = \begin{bmatrix} \Lambda^{(m)}(-\bar{p}) \\ \vdots \\ \Lambda^{(m)}(\bar{p}) \end{bmatrix} \tag{6-2}$$

The system matrix is written as,

$$\mathbf{K}^{(m)} = jm\,\omega_S\,\xi \begin{bmatrix} \frac{Z(-\bar{p})}{-\bar{p}} \sum_{k'=0}^{\infty} \Lambda_{k'}^{(m)}(-\bar{p})\Lambda_{k'}^{(m)}(-\bar{p}) & \cdots & \frac{Z(\bar{p})}{\bar{p}} \sum_{k'=0}^{\infty} \Lambda_{k'}^{(m)}(-\bar{p})\Lambda_{k'}^{(m)}(\bar{p}) \\ \vdots & \vdots & \vdots \\ \frac{Z(-\bar{p})}{-\bar{p}} \sum_{k'=0}^{\infty} \Lambda_{k'}^{(m)}(\bar{p})\Lambda_{k'}^{(m)}(-\bar{p}) & \cdots & \frac{Z(\bar{p})}{\bar{p}} \sum_{k'=0}^{\infty} \Lambda_{k'}^{(m)}(\bar{p})\Lambda_{k'}^{(m)}(\bar{p}) \end{bmatrix} \tag{6-3}$$

Dividing (6-1) by $\lambda^{(m)}$ and summing over m, we get an equation for the azimuthal mode coupling.

$$\sum_{m'=-\infty}^{\infty} j^{-m'} \Lambda_p^{(m')} = \sum_{m'=-\infty}^{\infty} \mathbf{K}^{(m')}/\lambda^{(m')} \sum_{m'=-\infty}^{\infty} j^{-m'} \Lambda_p^{(m')} \tag{6-4}$$

Because of the denominator on the right side of the equation, it is clear that (6-4) is a dispersion relation type equation.

To find the relation between the dispersion equation (6-4) and the eigenvalue problem (2-12), we need the following equation,

$$\mathbf{K}^{(m)} = jm\,\omega_S\,\xi\,\mathbf{\Lambda}^{(m')^T}\mathbf{\Lambda}^{(m)}\,\mathbf{Z} \tag{6-5}$$

and

$$\Lambda_p^{(m)} = \mathbf{\Lambda}^{(m)^T}\alpha^{(m)} \tag{6-6}$$

Now we are ready to show the relation between the two approaches. Again, to be not overwhelmed by large dimension, we only consider the coupling between m and m'. The equation (2-12) is written as,

$$\begin{bmatrix} \alpha^{(m)} \\ \alpha^{(m')} \end{bmatrix} = \begin{bmatrix} \mathbf{M}^{(m,m)}/\lambda^{(m)} & \mathbf{M}^{(m,m')}/\lambda^{(m)} \\ \mathbf{M}^{(m',m)}/\lambda^{(m')} & \mathbf{M}^{(m',m')}/\lambda^{(m')} \end{bmatrix} \begin{bmatrix} \alpha^{(m)} \\ \alpha^{(m')} \end{bmatrix} \tag{6-7}$$

Multiplying by $[\, j^{-m}\Lambda^{(m)T} \quad j^{-m'}\Lambda^{(m')T} \,]$, using (4-3), (6-5) and (6-6), we get

$$
\begin{aligned}
&(j^{-m}\Lambda_p^{(m)} + j^{-m'}\Lambda_p^{(m')}) \\
&= [\, (\mathbf{K}^{(m)}/\lambda^{(m)} + \mathbf{K}^{(m')}/\lambda^{(m')}) j^{-m}\Lambda^{(m)T} \quad (\mathbf{K}^{(m)}/\lambda^{(m)} + \mathbf{K}^{(m')}/\lambda^{(m')}) j^{-m'}\Lambda^{(m')T} \,] \begin{bmatrix} \alpha^{(m)} \\ \alpha^{(m')} \end{bmatrix} \\
&= (\mathbf{K}^{(m)}/\lambda^{(m)} + \mathbf{K}^{(m')}/\lambda^{(m')}) (j^{-m}\Lambda_p^{(m)} + j^{-m'}\Lambda_p^{(m')}) \qquad (6\text{-}8)
\end{aligned}
$$

which is the same as the dispersion equation (6-4).

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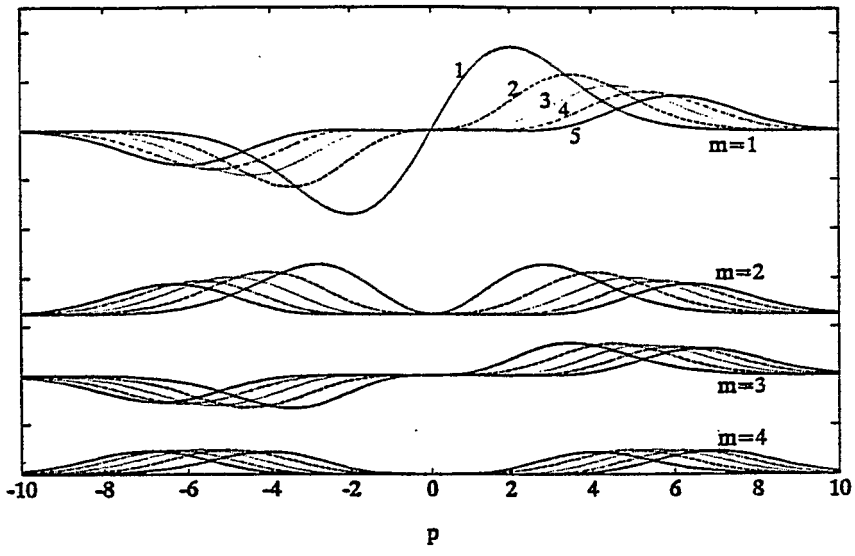

Fig.1. Hankel Spectra of Gaussian Distribution, Half Bunch Length 1 *rad*.

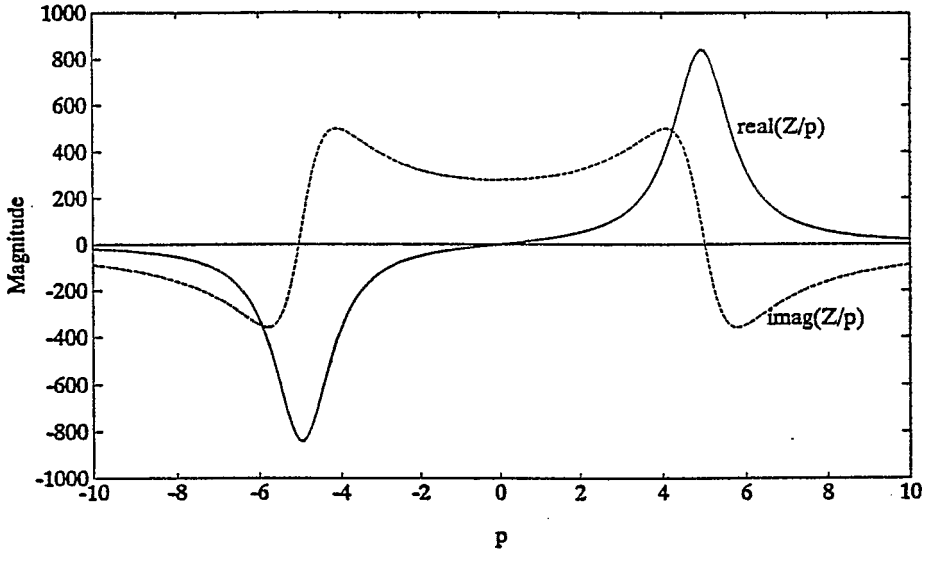

Fig.2. Resonator Impedance $Z(p)/p$, tuned at $p = 5$, with $Q = 3$.

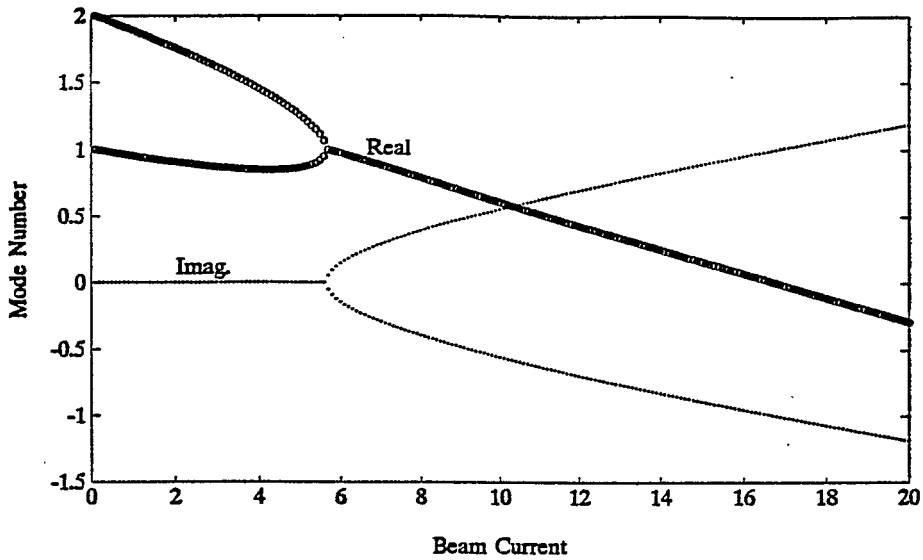

Fig.3. A Typical Mode Coupling.

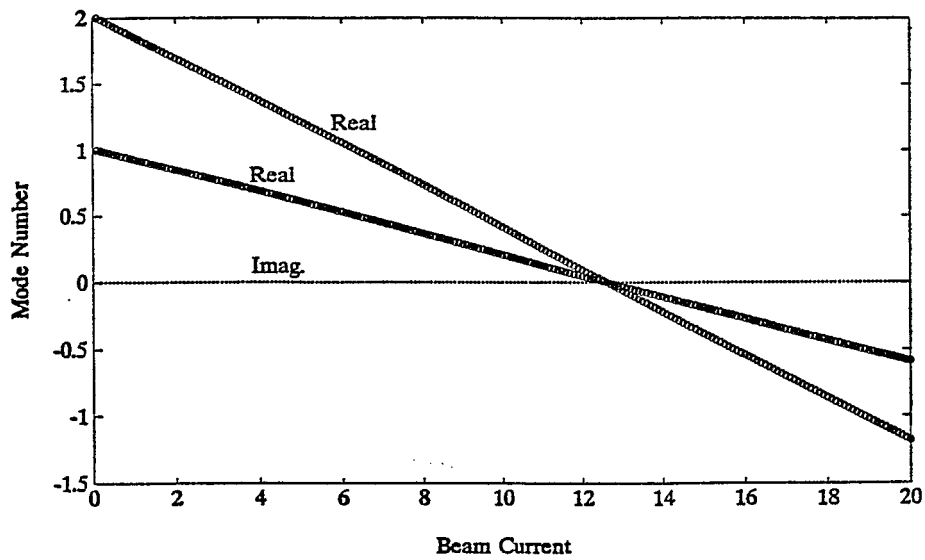

Fig.4. A Typical Mode Crossing.

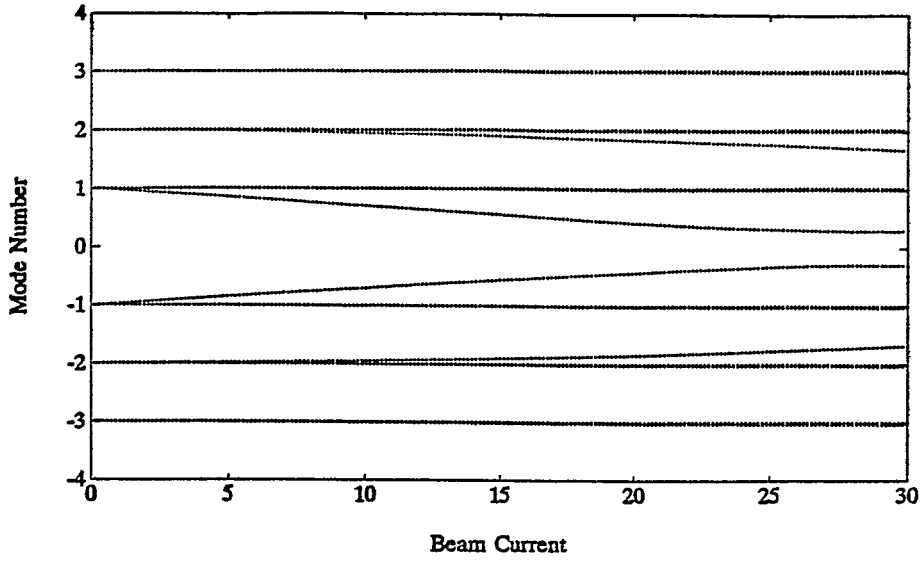

Fig.5a. A Mode Coupling including Azimuthal Modes -4 to 4.

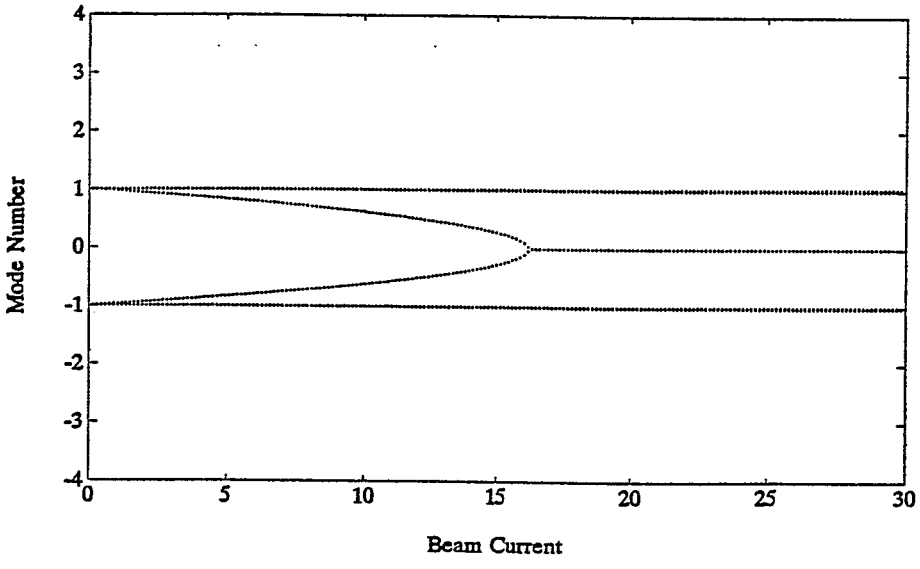

Fig.5b. Calculated with only Azimuthal Modes 1 and -1.

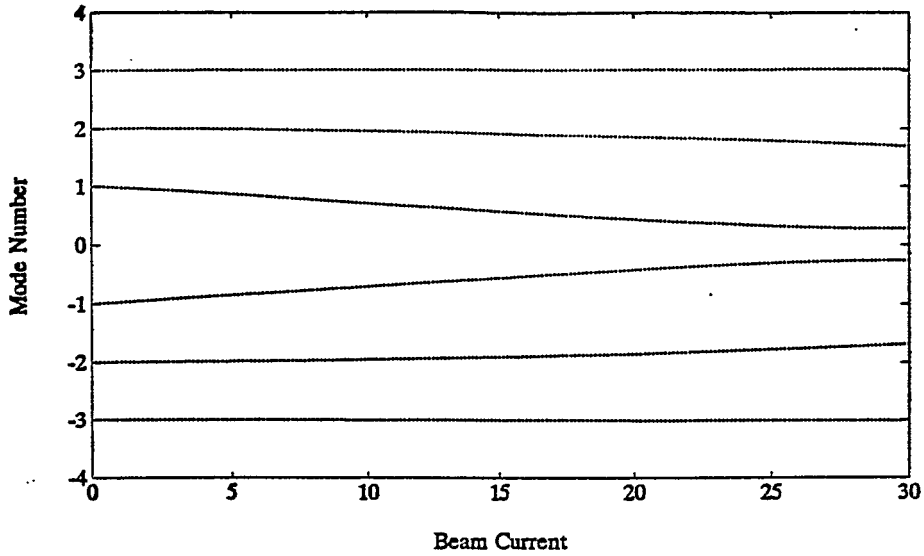

Fig.6a. Mode Coupling using Simplified Model, Cavity tuned at $p = 1$.

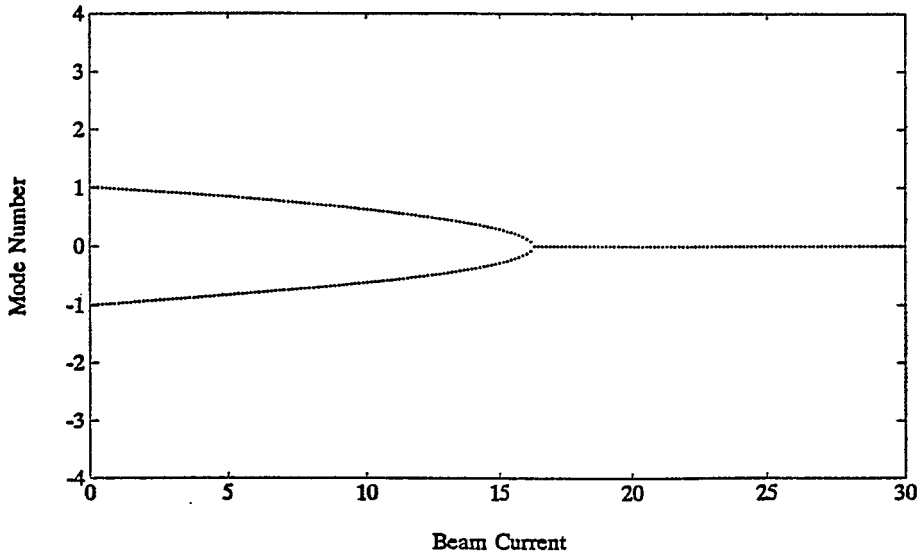

Fig.6b. Mode Coupling using Simplified Model, Cavity tuned at $p = 1$.

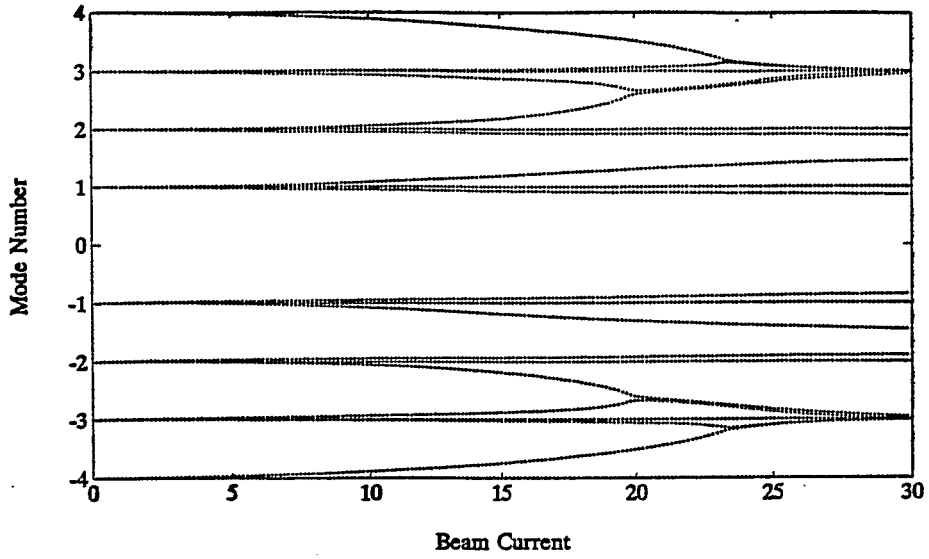

Fig.7a. Mode Coupling, Cavity tuned at $p = 5$.

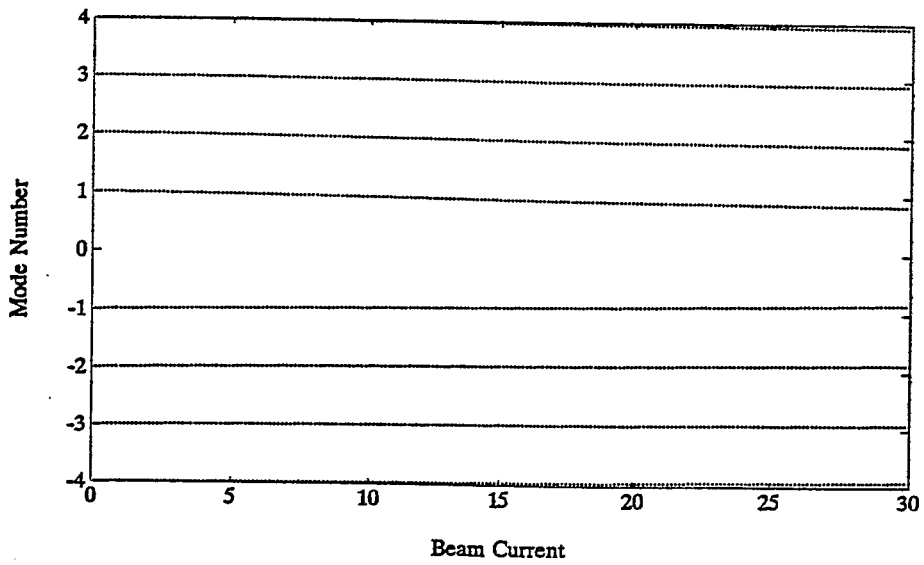

Fig.7b. Mode Coupling using Simplified Model, Cavity tuned at $p = 5$.

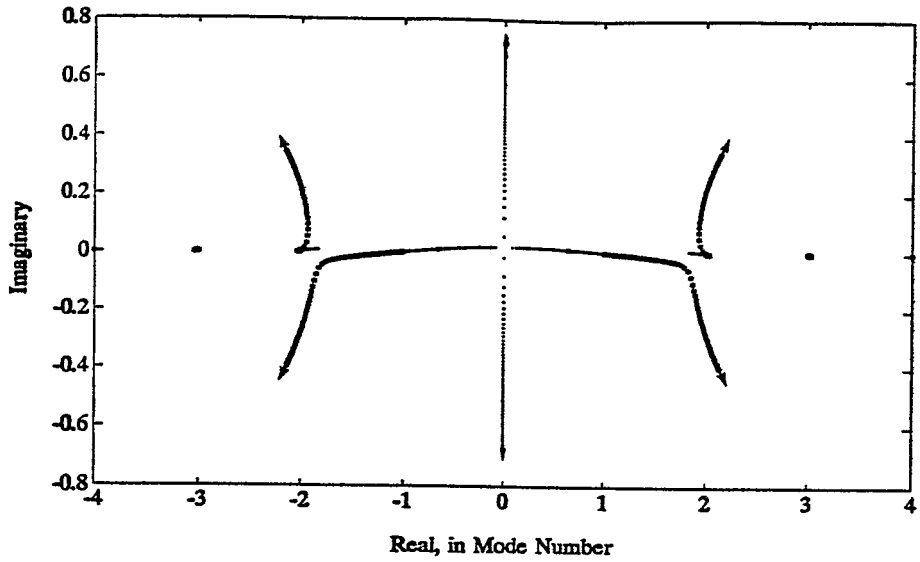

Fig.8a. Mode Coupling between Azimuthal Modes $m = \pm 1$.

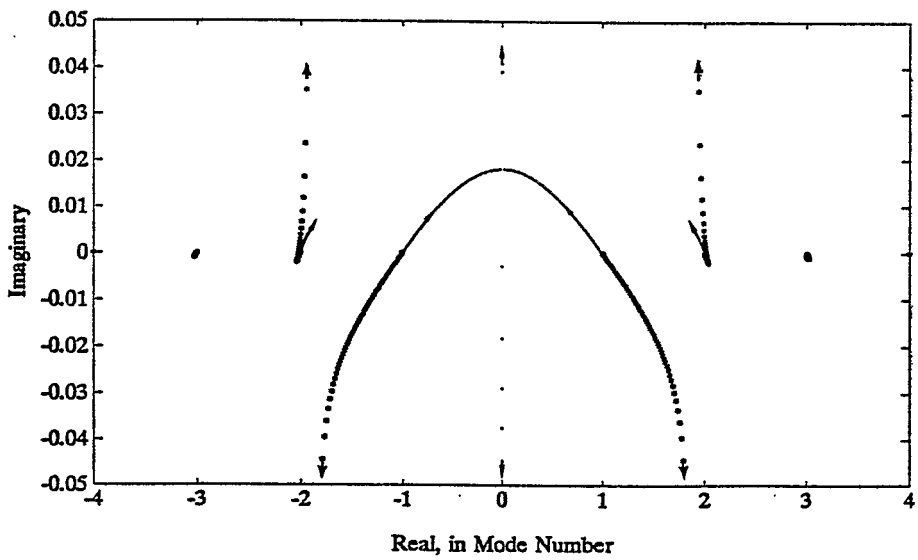

Fig.8b. Blow-Up of Fig.8a.

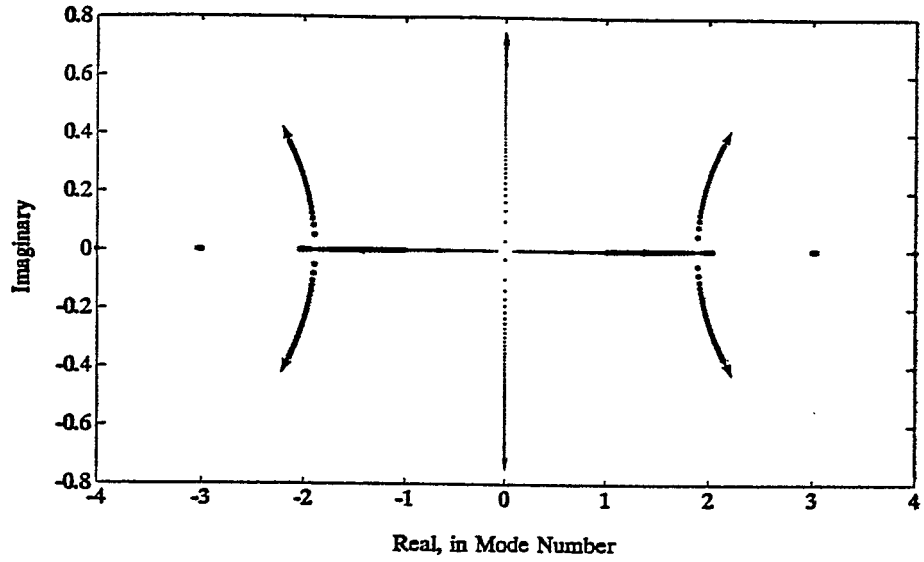

Fig.9a. A 'Pure' Mode Coupling.

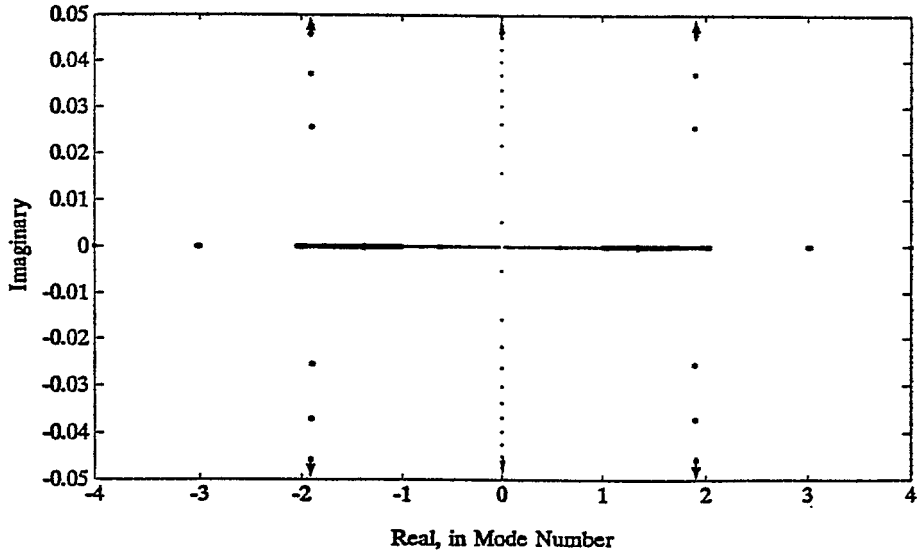

Fig.9b. Blow-Up of Fig.9a.

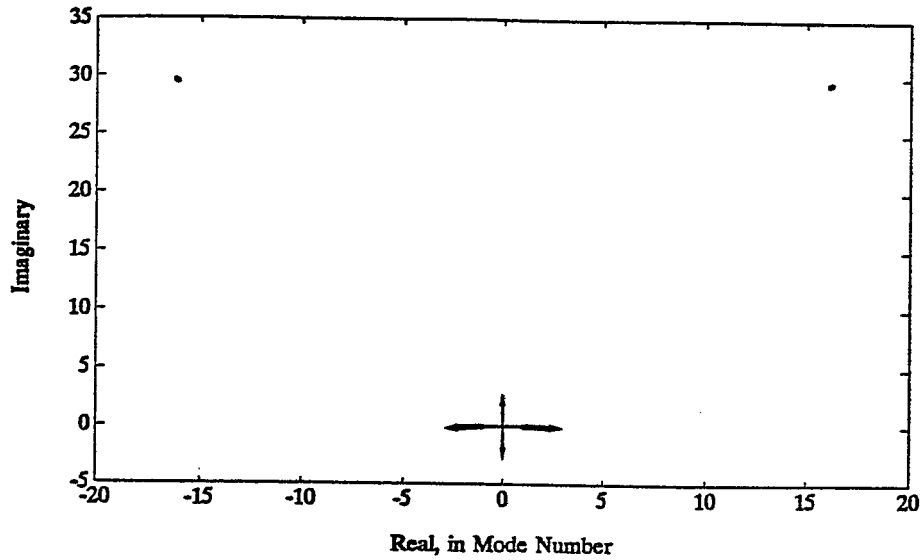

Fig.10a. Root Loci of Beam Loading using Robinson Approach.

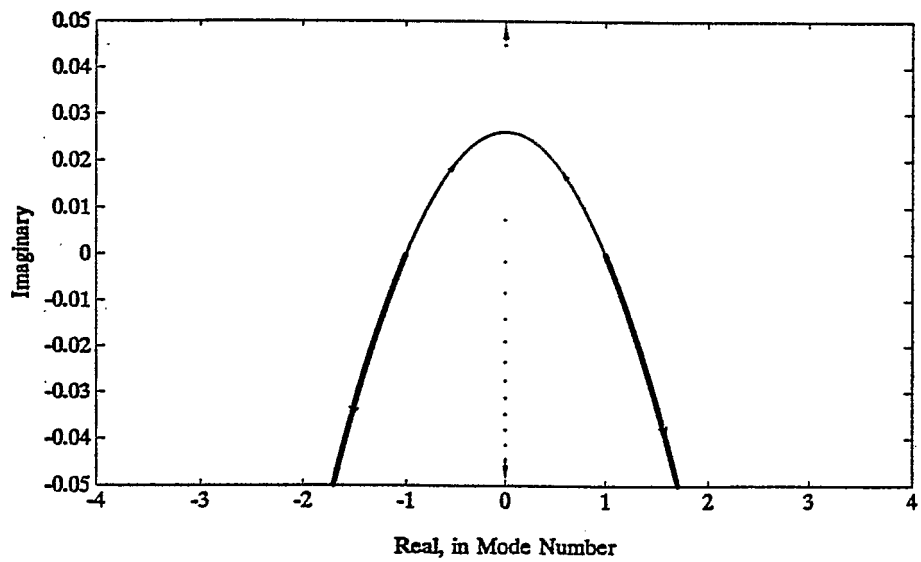

Fig.10b. Blow-Up of Fig.10a.