Saw-Tooth Instability as Nonlinear Mode Coupling

S. Heifets
Stanford Linear Accelerator Center, Stanford University, Stanford, CA 94309

Abstract

Dynamics of nonlinear oscillations is studied for of a single bunch above the threshold of microwave instability and for beam-beam interaction.

1 INTRODUCTION

There are number of observation of substantially non-linear coherent effects.

Some of them

1. **Beam-beam blow-up and O-type oscillations,**
2. **Beam-Ion instability**
3. **Saturation of single/multi-bunch instabilities,**
4. **Saw-tooth instability (SLC DR, P. Krejcik et.al.)**
5. **Relaxation oscillations in the beam interacting with high-Q resonator (SPEAR, J. Sebek, C. Limborg)**
6. **Transverse relaxation oscillations (K. Harkey, Argon).**

Similar problems exist and were studied in hadron machines (P.Colstoke).

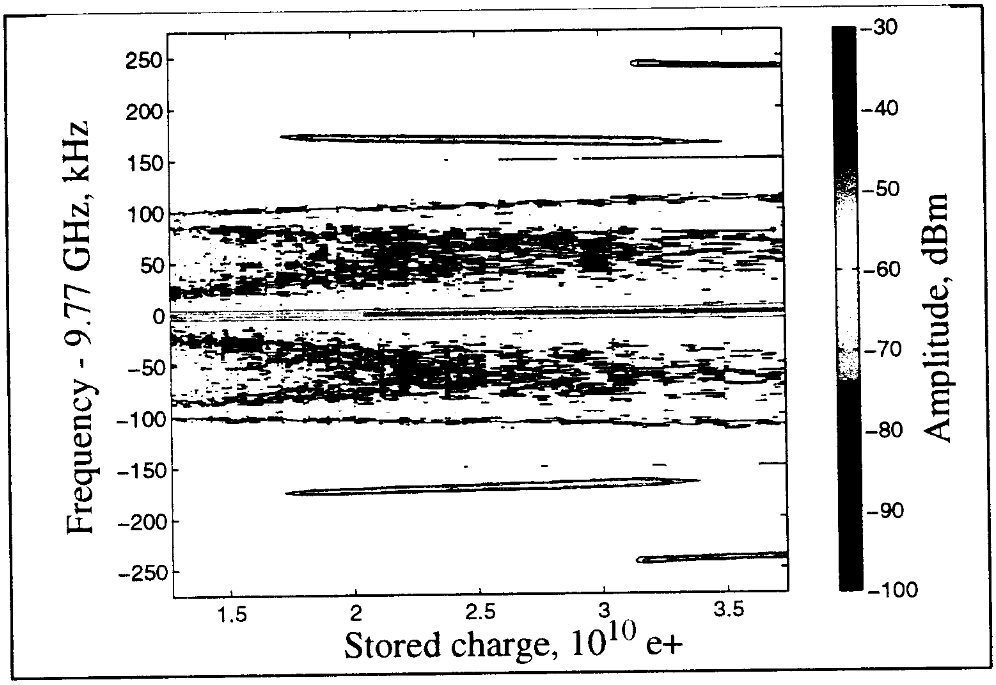
The 1D saw-tooth instability provide a simple case for study of these phenomena. It was observed in many laboratories and can be considered as onset of the microwave instability.

Beam-beam is another well-known nonlinear problem.

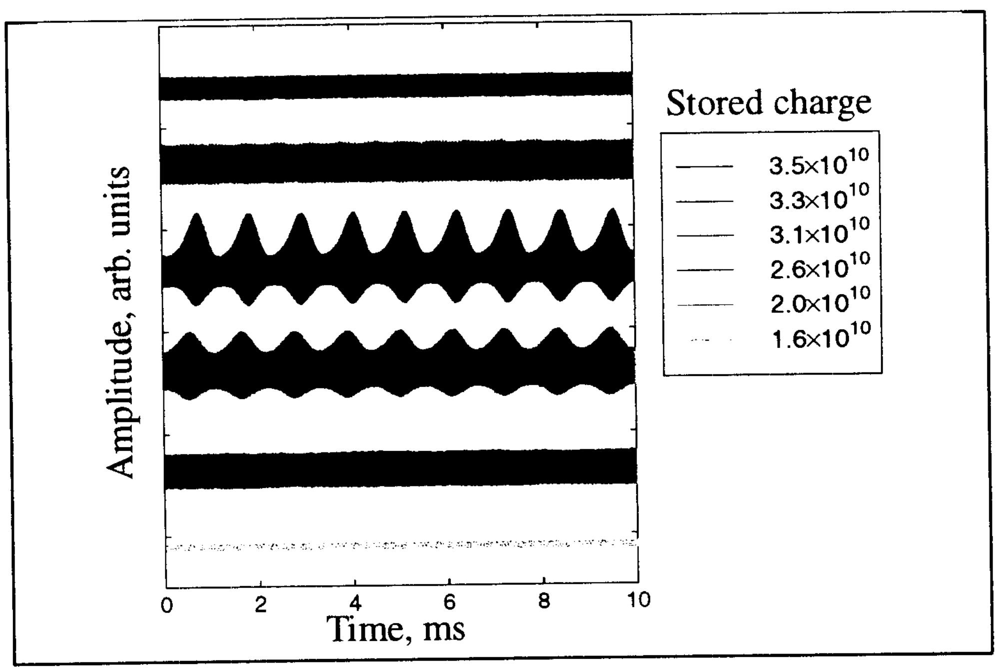
These problems are discussed below.

Typical Data for a Single Store

Spectrum Analyzer Data



Scope Traces for Different Stored Charge



2 Notations and Haissinski Solution

*** Diffusion and radiation damping are described by the Fokker-Plank equation.**

*** In a steady-state, solution of the F-Pl is (Haisinskii, 1973)** $\rho_H(J) = (1/Z_H)e^{-H_H(J,\rho)}$

*** Several minima of SCP at large** λ **are possible CMI and Baartman-Dyachkov mechanism (1995) then are important**

*** In this note we consider relatively small** λ, $U_H(x)$ **has one minimum, coherent frequency shift is small.**

*** In the time-dependent case,**

$$\rho(J,\phi,s) = \rho_H(J) + \sum_n \rho_n(J)e^{in\phi}, \quad H = H_H(J) + \sum_n U_n(J,s)e^{in\phi}, \tag{1}$$

. where

$$U_n(J,s) = \lambda \int dJ'd\phi' \rho_m(J,s)R_{m,m'}(J,J'), \quad \lambda = \frac{N_b r_0}{2\pi R\gamma\alpha\delta_0^2} \tag{2}$$

$$R_{m,m'}(J,J') = -\frac{4\pi}{Z_0}\int \frac{d\omega}{2\pi i}\frac{Z(\omega)}{\omega}C_m(J,\omega)C_{m'}^*(J',\omega), \tag{3}$$

$$C_m(J,\omega) = \int \frac{d\phi}{2\pi}e^{-im\phi}e^{ix(J,\phi)}. \tag{4}$$

where $x(J,\phi)$ **is particle trajectory in the Haisinskii potential.**

3 Fokker-Plank equation. Linear Approximation

*** The Fokker-Plank equation defines azimuthal harmonics ρ_n**

*** In the linear approximation, all azimuthal modes are independent and are the superposition of radial modes**

$$\rho_n(J, s) = \rho'_H(J) \sum_\nu b_\nu X_\nu(J) e^{-i(\omega_H - \bar{\omega})s} \tag{5}$$

$$\int dJ' M_m(J, J') X_\nu(J') = -\nu X_\nu(J). \tag{6}$$

$$M_m(J, J') = 2\pi\lambda R_{mm}(J, J')\rho'_H(J') - \delta(J - J')(\omega_H(J) - \bar{\omega}), \tag{7}$$

*** The beam is linearly unstable if at least one of the eigen-values ν has positive imaginary part $\Gamma_\nu = Im[\nu] > \gamma_n$.**

*** If $\rho'_H(J)$ is monotonic function, the beam stability (apart of Landau damping) depends on the anharmonicity of the trajectories and given by the asymmetric part of R_{mm} [Oide].**

*** The structure of the modes is**

$$X_\nu(J) = \frac{r_m(J)}{\omega_H(J) - \bar{\omega} - \nu}, \tag{8}$$

where $r_m(J)$ is a smooth function of J.

Mode is localized around the resonance value J_r, $\omega(J_r) = \bar{\omega} + \Omega$ with the width Γ_m.

4 Comparison with a nonlinear oscillator

* Above the threshold, mode interaction may become important.

When there are many unstable modes $\rightarrow$ turbulent regime of the microwave instability.

* What can be expected when a single unstable mode is dominant?

* Quasi-linear approach (O'Neil, 1965; Yongho Chin, Yokoya, 1984)

Result: Particle Trapping and saturation of the instability.

* Quasi-steady state: Shonfeld (1985), Meller (1986). However, it is difficult to use in time-domain self-consistent calculations

* Analogy with nonlinear oscillator + periodic external perturbation given by the growing mode.

$$H(J, \phi, s) = H_0(J) + \eta \cos(\phi - \Omega s - \psi_0), \tag{9}$$

$\Omega = \omega_0 + \Delta$, $\eta = \frac{\epsilon}{2}\sqrt{\frac{2J}{\omega_0}}$.

* Resonance island exists at arbitrary small amplitudes ϵ.

The width of the resonance depends on ϵ/κ and can be large even for small ϵ.

4.1 Can we get these results from the Liouville equation?

Neglect self-consistency. Consider dipole $U_1 = U_{-1}^* = \eta\sqrt{J}e^{-i\Omega s}$, η may grow $\eta = \eta_0 e^{\Gamma s}$.

In the linear approximation,

$$\rho_{res} = \frac{\eta\rho_H'(J)\sqrt{J}}{(J - J_r - iw)}e^{-i\Omega s}, \tag{10}$$

*** The linearized equation describes the resonance mode but not modification of the rest of the phase plane.**

*** Include now coupling to harmonics ρ_0 and $\rho_{\pm 2}$**

$$\dot{\rho}_1 + i\omega_H\rho_1 - i\rho_H' U_1 + 2i\frac{\partial U_1^*}{\partial J}\rho_2 - i\frac{\partial \rho_0}{\partial J}U_1 + i\frac{\partial \rho_2}{\partial J}U_1^* = 0. \tag{11}$$

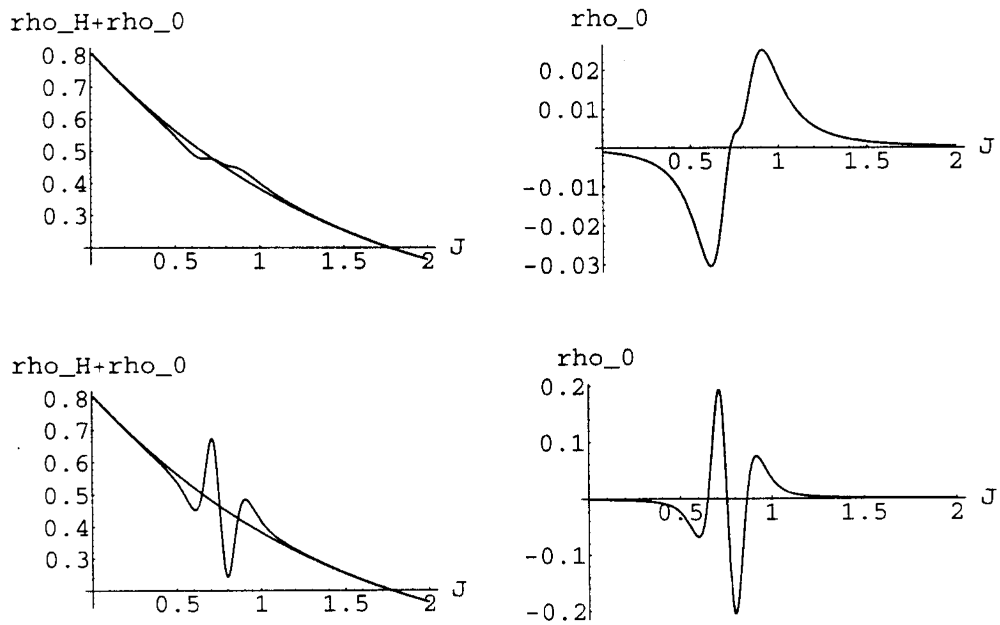
*** Equations for ρ_0 and ρ_2 can be solved explicitly.**
Introduce $f(J)$, $\bar{\rho}_1 = \eta\sqrt{J}f$.

$$(J - J_r - iw)f = \rho_H'(J) + \frac{i\eta^2}{2w}\frac{\partial^2}{\partial J^2}[Jf^* - c.c.] - \{\frac{\eta^2(\partial f/\partial J)}{J - J_r - iw} + \frac{\partial}{\partial J}[\frac{\eta^2 J(\partial f/\partial J)}{J - J_r - iw}]\}. \tag{12}$$

The first term and the third terms give corrections due to coupling to ρ_0 and ρ_2.

*** For small η, equation can be solved by iterations.**
Corrections are small if $(J - J_r)^2 > \eta\sqrt{J_r}$
i.e. outside of the separatrix.

This suggest that interaction of two modes may be essential.



Distribution function (m = 0) pertubed by dipole (m = 1, above) and quadrupole (m = 1, below) modes

$\eta = 0.03$

$w = 0.2$

$J_r = 0.75$

$\varkappa = 0.1$

$\omega_H = 0.7 + 0.1\,J$

"shape0.eq.eps"

5 Nonlinear Regime

Keep harmonics $m = 0, 1, 2$.

Expand

$$\rho_1(J,s) = \rho'_H(J) b_\nu X_\nu(J) e^{-i(\omega_H - \bar{\omega})s} \tag{13}$$

$$\rho_2(J,s) = \rho'_H(J) a_\mu Y_\mu(J) e^{-2i(\omega_H - \bar{\omega})s} \tag{14}$$

$$M_m(J,J') = 2\pi\lambda R_{mm}(J,J')\rho'_H(J') - \delta(J-J')(\omega_H(J) - \bar{\omega}), \quad m = 1,2 \tag{15}$$

$$\int dJ' M_1(J,J') X_\nu(J') = -\nu X_\nu(J), \tag{16}$$

$$\int dJ' M_2(J,J') Y_\mu(J') = -\mu Y_\mu(J). \tag{17}$$

The radial amplitudes satisfy equations

$$\dot{b}_\nu + (i\nu + \gamma_d) b_\nu + i \sum_{\mu,\sigma} C^\nu_{\mu,\sigma} a_\mu b^*_\sigma + i \sum_\sigma d_{\nu,\sigma} b_\sigma = 0, \tag{18}$$

$$\dot{a}_\mu + (2i\mu + \gamma_q) a_\mu + i \sum_{\sigma,\lambda} g^\mu_{\sigma,\lambda} b_\lambda b_\sigma + i \sum_\sigma f_{\mu,\sigma} a_\sigma = 0, \tag{19}$$

$\frac{\partial \rho_0}{\partial s}$ **gives another set of equations:**

$$\dot{d}_{\nu\sigma} + \gamma_0 d_{\nu\sigma} = -i(\nu'^* - \sigma') P^{\nu,\sigma}_{\nu',\sigma'} b^*_{\nu'} b_{\sigma'} - 2i(\mu'^* - \lambda') Q^{\nu,\sigma}_{\mu',\lambda'} a^*_{\mu'} a_{\lambda'}, \tag{20}$$

$$\dot{f}_{\mu\lambda} + \gamma_0 f_{\mu\lambda} = (\nu'^* - \sigma') F^{\mu,\lambda}_{\nu',\sigma'} b^*_{\mu'} b_{\sigma'} + 2(\mu'^* - \lambda') G^{\mu,\lambda}_{\mu',\lambda'} a^*_{\mu'} a_{\lambda'}. \tag{21}$$

Coefficients P, Q, F, **and** G **are constants. For example,**

$$P^{\nu,\sigma}_{\nu'\sigma'} = -\int dJ \frac{\partial}{\partial J}[\frac{\omega_H - \bar{\omega} - \sigma}{\rho'_H} \bar{X}_\nu X_\sigma] \frac{\partial}{\partial J}[\rho'_H X^*_{\nu'} X_{\sigma'}]. \tag{22}$$

6 Single Mode

Consider single unstable radial dipole mode $\nu" \equiv Im[\nu] > 0$ taking into account coupling to ρ_0.

$$\dot{b} + (i\nu + \gamma_d)b + idb = 0, \quad \dot{d} + \gamma_0 d = -2\nu" P|b|^2. \tag{23}$$

Here $P = P^{\nu,\nu}_{\nu,\nu}$.

In the nonlinear regime, d, the momentum of ρ_0, modifies the linear coherent frequency ν and can stop and even reverse the sign of the growth rate.

FP: $x = y = 0$, stable if $\nu" > \gamma_d 0$,
FP: A new FP exists and stable if $\nu" > \gamma_d 0$.

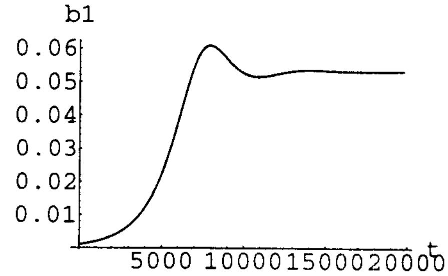
The non-trivial FP corresponds to a limiting cycle where $b = Be^{i\Omega s}$ with real Ω, $\dot{d} = const = i\gamma_d - \nu$.

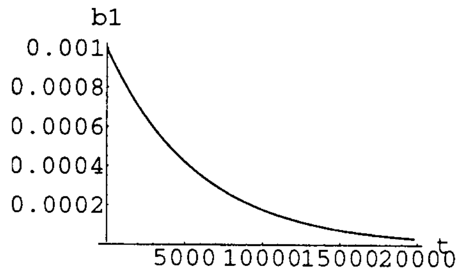
$$\Omega = -Re(\nu) + (Im[\nu] - \gamma_d)\frac{Re[P]}{Im[P]}, \quad |b|^2 = \frac{\gamma_0}{2Im[P]}(1 - \frac{\gamma_d}{Im[\nu]}). \tag{24}$$

This is the main result of quasi-linear theory:

ρ_0 is distorted by the unstable mode in such a way that the mode is stabilized.

A constant distortion changes both rms bunch length and the energy spread.





lambda = 6.88; mu = 1.5; alpha = 0.4;
Above : Limiting Cycle, gamdip = 0,
mu "=6.27 10^(-4);
Sqrt[gam0/2Im[P]]=0.053; b[0]=10^(-3)

Below : Stability, gamdip = 8. 10^(-4)

7 Two interacting modes

Analysis in more general cases is complicated.

Let us consider two dipole modes, one unstable with eigenvalues ν, $Im[nu] = \nu" > \gamma_d$, and another stable mode with eigenvalue μ.

$$\dot{b}_\nu + (i\nu + \gamma_d + id_{\nu,\nu})b_\nu + id_{\nu,\mu}b_\mu = 0, \tag{25}$$

$$\dot{b}_\mu + (i\mu + \gamma_d + id_{\mu,\mu})b_\mu + id_{\mu,\nu}b_\nu = 0, \tag{26}$$

and four equations for d, for example

$$\dot{d}_{\nu\mu} + \gamma_0 d_{\nu\mu} = -i \sum_{\nu',\mu'} (\nu'^* - \mu') P^{\nu,\mu}_{\nu',\mu'} b^*_{\nu'} b_{\mu'}. \tag{27}$$

* The mode stability depends on $\Gamma_\nu = \gamma_d - Im[\nu + d_{\nu,\nu}]$.

Consider small $b[0]$, $d[0]$. The unstable mode leads to growth of $d_{\nu,\nu}$:

$\dot{d}_{\nu\nu} + \gamma_0 d_{\nu\nu} = -2\nu" P^{\nu,\nu}_{\nu,\nu} |b_\nu|^2$.

If $Im[P^{\nu,\nu}_{\nu,\nu}] > 0$, then $d_{\nu\nu} < 0$ and mode can be stabilized and even starts to decay with time.

* Due to the same mechanism, Γ_μ of stable mode is modified.

If $Im[P^{\mu,\mu}_{\nu,\nu}] < 0$, the linearly stable mode can become unstable when linearly unstable mode saturates.

* After that, their roles interchange and the process can repeat itself.

* The fastest growing mode is the most stable mode in the linear approximation.

* Analysis gives the FP allowing the limiting cycles, $b_\nu \propto b_\mu \propto e^{-i\Omega s}$

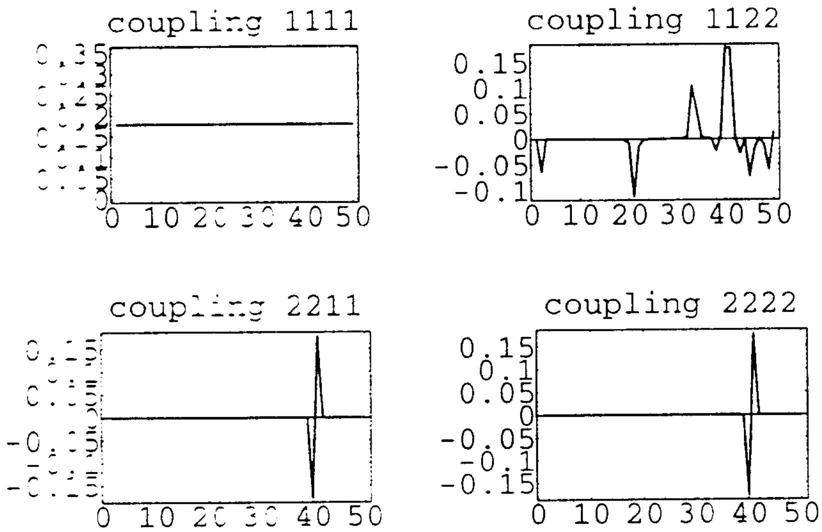
were Ω is real.

* Location of the FP can be determined analytically.

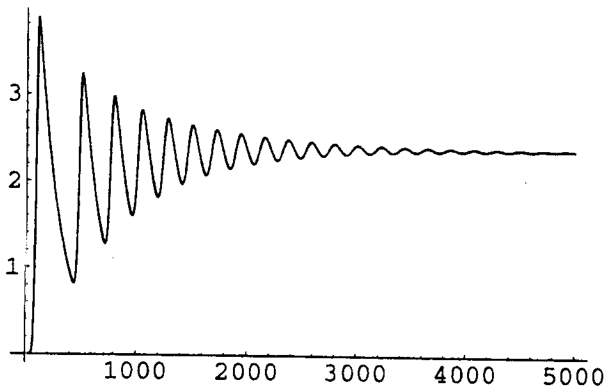
Computer simulations based on Eqs. () confirmed these results.

In the same way interaction of other modes can be explored. Fig. depicts saturation of two linearly unstable modes, one quadrupole and another dipole.

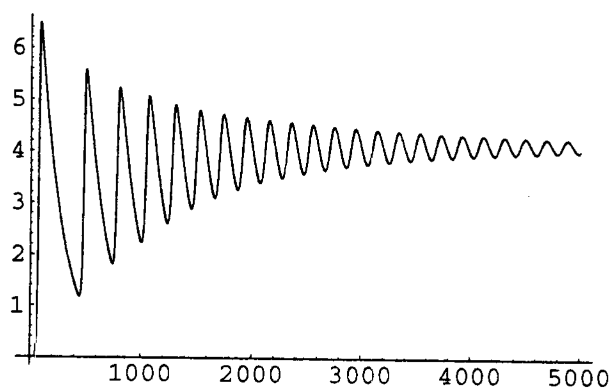
Fig. shows supression of the dipole mode in the interaction of linearly unstable and stable dipole modes with linearly unstable quadrupole mode.



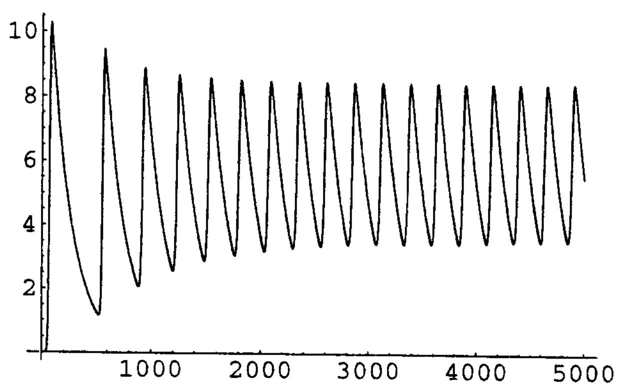
Coupling coefficients Im[P] for all 50 modes.
Unstable mode is #41, the most stable mode #40.



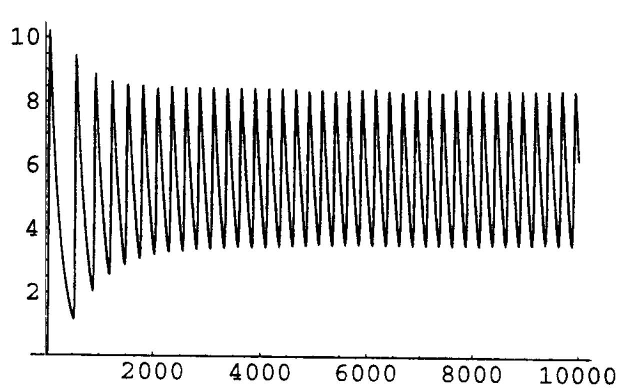
lambda=12.5664 alph=0.5 om'=0.1 om2p=0. jbar=0.5

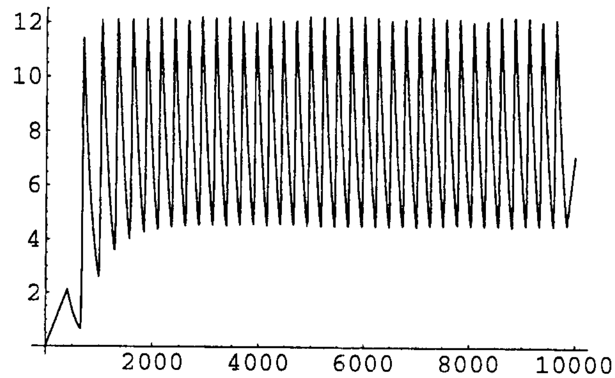


lambda=13.823 alph=0.5 om'=0.1 om2p=0. jbar=0.5

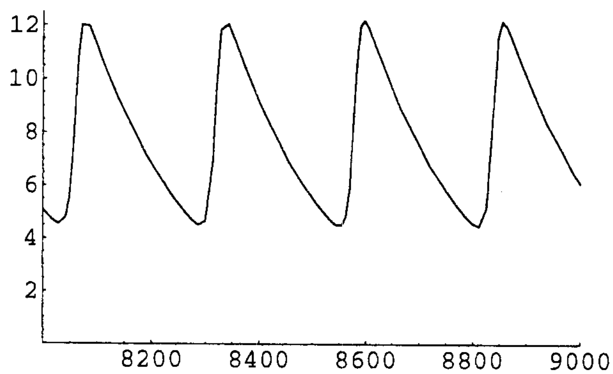


lambda=15.0796 alph=0.5 om'=0.1 om2p=0. jbar=0.5

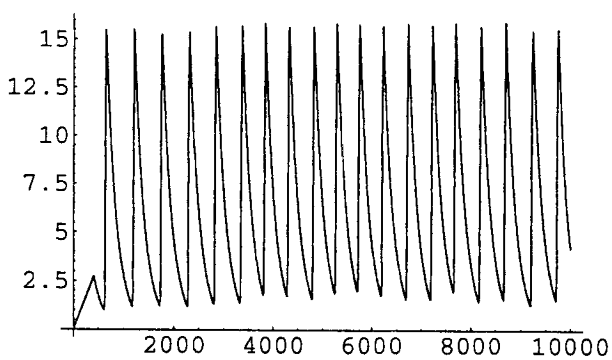




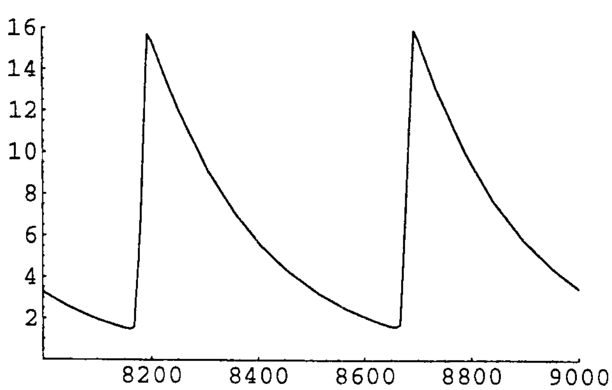
lambda=2.4 alph=0.5 om'=0. om2p=0.1 jbar=0.5

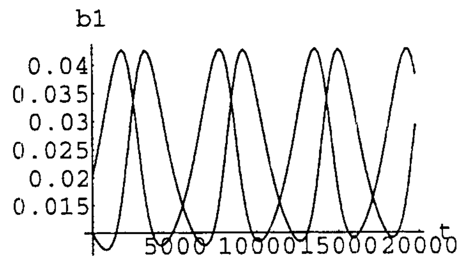


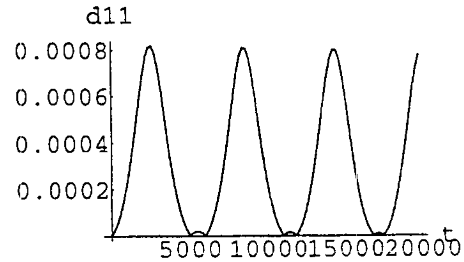
lambda=2.4 alph=0.5 om'=0. om2p=0.1 jbar=0.5



lambda=2.6 alph=0.5 om'=0. om2p=0.1 jbar=0.5





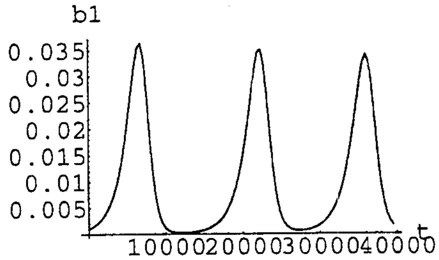


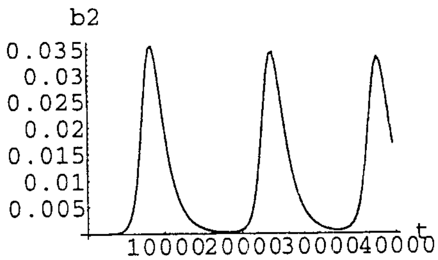
Code: relaxdipole .nb, lambda=6.88447 alpha=0.4 mu=1.5

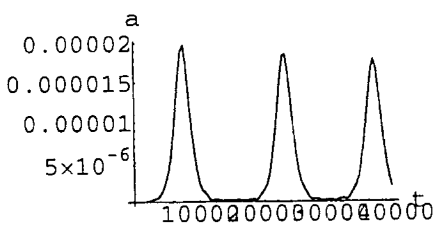
omega=Hass, gam0=0.00001
, gamdip=0., 2 dipole modes=unstable+stable, interaction through rho0,
b1[0]=2 10^(-2), b2[0]=10^(-2), d[ik][0]=0.0, eigen values=0.0429747 + 0.000627594 I 0.198057

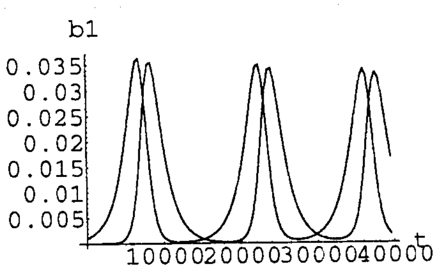
12

lam=6.88,mu=1.5,al=0.4









"oscenu15.eps"

$2d + 1q$, q = stabl

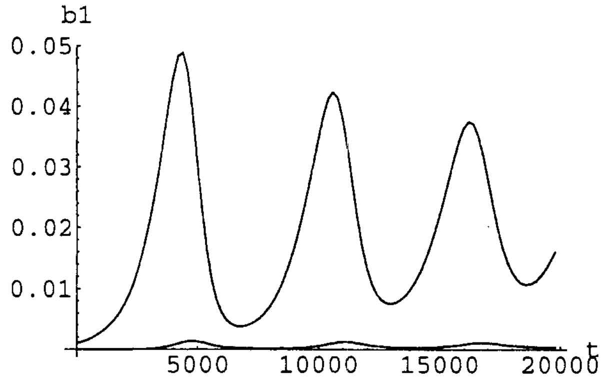
$\gamma_d = 10^{-5}$, $\gamma_\alpha = 10^{-5}$

$\gamma_q = 10^{-4}$

$b_1(0) = 10^{-3}$, dice b_2 = f

$\nu_1^{I} = 0.073$, $\nu_1^{II} = 6.2 \cdot 10^{-4}$, $\nu_2 = 0.15$

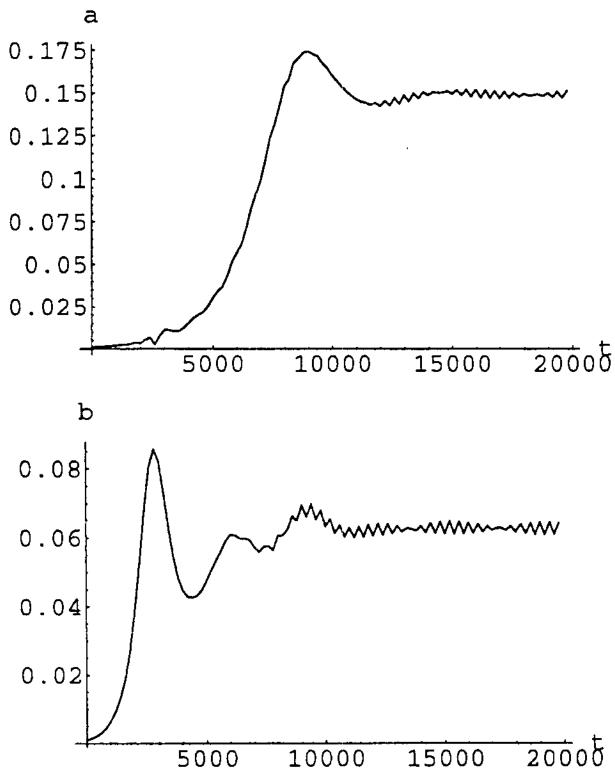
12



Out[65]= - Graphics -

Amplitude of linearly unstable dipole mode (above)
and dynamic decrement (below) due to coupling to rho_ 0.
When tha latter is larger than linear increment (0.0019)
growth reverses sign.

"sfeb2d17.eps"



Out[58]= - Graphics -

lambda = 6.88447 alpha = 0.9 mu = 1.5

gam0 = 0.001 gamdip = 0.0001 gamquad = 0.0001

Dipole + Quadrupole modes, both
linearly unstable. Stabilization is due to nonlinear interaction.

* rel dip q. sat. eps "

/.

8 Beam-Beam

1D beam-beam interaction can be analyzed in the same way.

$$H = Q_y J + \frac{2N_b^{(2)} r_0}{2\pi\gamma_1} \sum_{n,m,l} e^{i(n\psi+m\phi)} \int dJ' \rho_l^{(2)}(J',\phi) S_{n,l}^{(12)}(J,J'), \tag{28}$$

$$S_{n,l}^{1,2}(J,J') = -\int_0^\infty \frac{dt}{t} J_n[t\frac{2w_1^*\sqrt{J}}{\Sigma_x}] J_l[t\frac{2w_2^*\sqrt{J'}}{\Sigma_x}] e^{t^2}[1-\Phi[t]], \tag{29}$$

where J_n are Bessel functions, and $\Phi[t]$ is error function.

Assume that the closest resonance is $m + nQ_{1y} - lQ_{2y} = \Delta$, $\Delta- > 0$ and average fast oscillations.

Functions f,

$$\rho_n^{(1)} = f_n^{(1)} e^{-inQ_1^0 + i\Delta/2)t} \frac{\partial \rho_H^{(1)}(J)}{\partial J}; \quad \rho_l^{(2)} = f_l^{(2)} e^{-ilQ_2^0 - i\Delta/2)t} \frac{\partial \rho_H^{(2)}(J)}{\partial J}, \tag{30}$$

satisfy two coupled Vlasov equations.

They can be expanded in eigen-functions $V = (X, Y)$ of the matrix $[(0, M_1), (M_2, 0)]$ where

$$M_1(J,J') = \lambda_1 n \frac{\partial \rho_H^{(2)}(J)}{\partial J} S_{nl}^{1,2}(J,J') - \delta(J-J')[n(Q_1(J) - Q_1^0) + \Delta/2], \tag{31}$$

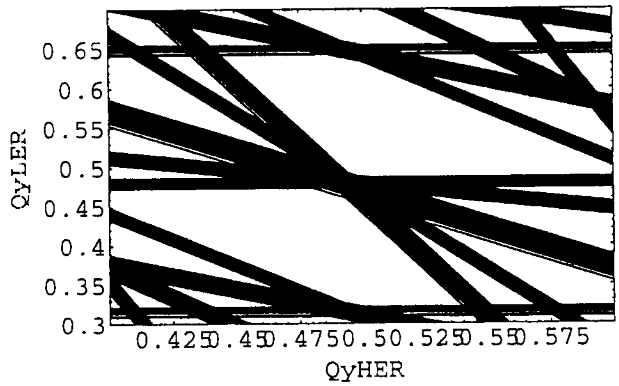
M_2 is obtained by exchange indexes.

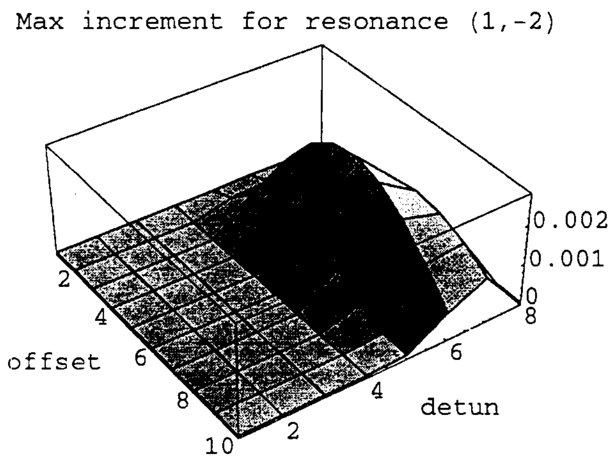
Nonlinear coupling through ρ_0 leads to equations

$$\dot{b}_\nu + i[\nu - i\gamma_0] b_\nu = i \sum_\mu d_{\mu,nu} b_\mu, \tag{32}$$

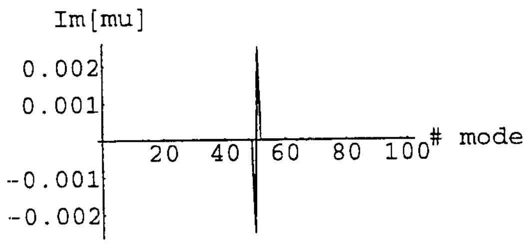
$$\dot{d}_{\nu\mu} + \gamma_d d_{\nu\mu} = i P_{\lambda\sigma}^{\nu\mu} b^* \lambda b_\sigma. \tag{33}$$

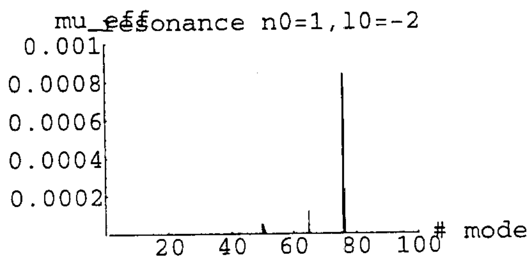
Preliminary result is shown in Fig.

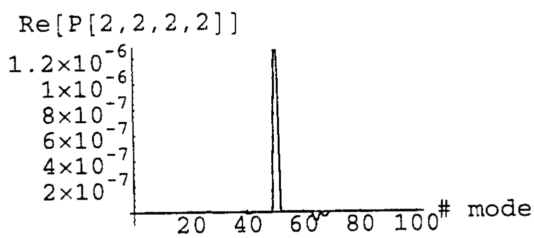

0.65
0.6
0.55
0.5
0.45
0.4
0.35
0.3
QyLER
0.425
0.45
0.475
0.5
0.525
0.55
0.575
QyHER

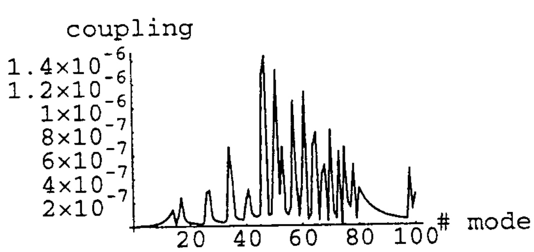



```
detun = -0.1 + i 0.01; i = 1, .. 8;
offy = [0.1 + (j - 1) 0.2 ] sigler, j = 1. .10;
```

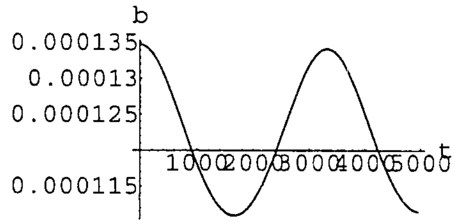

Im[mu]
0.002
0.001
-0.001
-0.002
20 40 60 80 100
mode

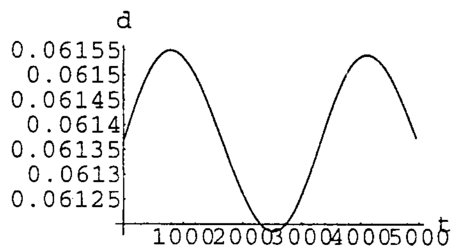

mu_resonance n0=1,l0=-2
0.001
0.0008
0.0006
0.0004
0.0002
20 40 60 80 100
mode


Re[P[2,2,2,2]]
1.2×10^-6
1×10^-6
8×10^-7
6×10^-7
4×10^-7
2×10^-7
20 40 60 80 100
mode

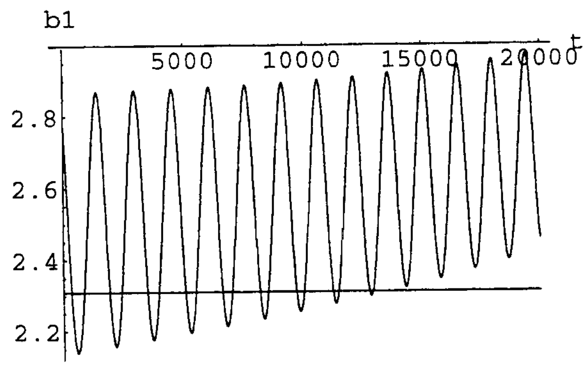

coupling
1.4×10^-6
1.2×10^-6
1×10^-6
8×10^-7
6×10^-7
4×10^-7
2×10^-7
20 40 60 80 100
mode

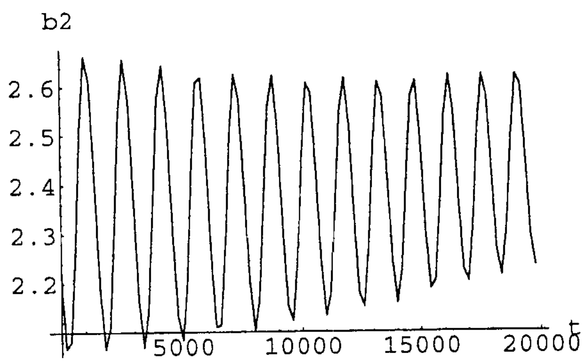
```
Show[GraphicsArray[{{%279}, {%280}}]]
```

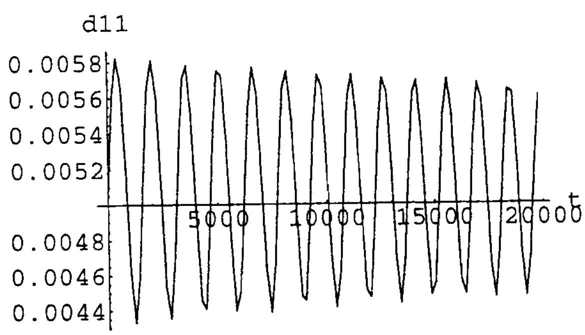




- GraphicsArray -


b1
5000
10000
15000
20000
t
2.8
2.6
2.4
2.2


b2
2.6
2.5
2.4
2.3
2.2
5000
10000
15000
20000
t


d11
0.0058
0.0056
0.0054
0.0052
5000
10000
15000
20000
t
0.0048
0.0046
0.0044

9 Conclusion

* Dynamics of the system in the nonlinear regime above the threshold of instability may be quite complicated and substantially depends on the impedance, radiation damping, and beam current.

* Additional to already known mechanisms of linear mode coupling and Baartman-Dyachkov mechanism, there is another mechanism of nonlinear mode coupling.

* We explore the last mechanism and demonstrated that it may lead to the saw-tooth oscillations for beam current close and above the threshold of microwave instability.

* Results are similar to that obtained by Stupakov, Breisman, Pekker (1996) but, from our point of view, allows more systematic approach to the nonlinear collective phenomena.

Application of this approach to other cases mentioned in the introduction, hopefully, will be presented later.