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The Maximum x and Maximum y in the Presence of Linear Coupling

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1. Introduction

This note provides analytical results for the following 2 problems:

1. In the presence of linear coupling, given the initial coordinates x_0, x_0', y_0, y_0' at the longitudinal position s_0, what is the maximum x and the maximum y at the position s?
2. In the presence of linear coupling, given the initial total emittance ϵ_t. What is the maximum x and the maximum y at the position s?

2. The Maximum x and y for a given x_0, x_0', y_0, y_0'

I assume that the motion is linearly coupled by a skew quadrupole field. The normal modes and the parameters β, α, γ of each normal mode can be found using the results of Edwards and Teng.[1] The uncoupled normal mode coordinates v, p_v, u, p_u are related to x, x', y, y' by a 4×4 matrix R

$$x = R\, v \tag{2.1}$$

The R matrix can be computed from the one turn transfer matrix.[1] The R matrix can be written as[1]

$$R = \begin{pmatrix} I\cos\varphi & \overline{D}\sin\varphi \\ -D\sin\varphi & I\cos\varphi \end{pmatrix}, \tag{2.2}$$

where D, $\overline{D}$ are 2×2 matrices, $\overline{D} = D^{-1}$, $|D| = 1$ and I is the 2×2 identity matrix. D and φ can be found[1] from the one turn transfer matrix.

For the normal mode v, p_v the parameters β_1, α_1, γ_1, ca be found,[1] such that

$$\epsilon_1 = \gamma_1 v^2 + 2\alpha, v p_v + \beta_1 p_v^2, \tag{2.3a}$$

is an invariant. For the u, p_u mode, one finds the parameters β_2, α_2, γ_2 such that

$$\epsilon_2 = \gamma_2 \mu^2 + 2\alpha_2 \mu p_\mu + \beta^2 p_\mu^2 \tag{2.3b}$$

is an invariant.

For the given initial coordinates x_0, x_0', y_0, y_0' at s_0, one can find ϵ_1 and ϵ_2 by using Eq. (2.1) to find the corresponding normal mode coordinates, v_0, p_{v_0}, μ_0, p_{μ_0} and then using Eqs. (2.3) to find the invariants ϵ_1, ϵ_2.

To find x as a function of s, one notes that the v and u motion is given by[1]

$$\begin{aligned}
v &= (\beta_1 \epsilon_1)^{\frac{1}{2}} \cos \psi_1 \\
p_v &= (\gamma_1 \epsilon_1)^{\frac{1}{2}} \cos (\psi_1 - \delta_1) \\
u &= (\beta_2 \epsilon_2)^{\frac{1}{2}} \cos \psi_2 \\
p_u &= (\gamma_2 \epsilon_2)^{\frac{1}{2}} \cos (\psi_2 - \delta_2) \\
\delta_1 &= \arctan (1/\alpha_1), \quad \delta_2 = \arctan (1/\alpha_2).
\end{aligned} \tag{2.4}$$

ψ_1, ψ_2 are the betatron phases of the normal modes.

The x motion can then be found using Eqs. (2.1) and (2.2),

$$x = v \cos \varphi + \left(\overline{D}_{11} u + \overline{D}_{12} p_u\right) \sin \varphi \tag{2.5}$$

$$\begin{aligned}
x = &(\beta_1 \epsilon_1)^{\frac{1}{2}} \cos \varphi \quad \cos \psi_1 \\
&+ \overline{D}_{11} (\beta_2 \epsilon_2)^{\frac{1}{2}} \sin \varphi \quad \cos \psi_2 \\
&+ \overline{D}_{12} (\gamma_2 \epsilon_2)^{\frac{1}{2}} \sin \varphi \quad \cos (\psi_2 - \delta_2),
\end{aligned}$$

$$\begin{aligned}
x = &(\beta_1 \epsilon_1)^{\frac{1}{2}} \cos \varphi \quad \cos \psi_1 \\
&+ \left(\overline{D}_{11} (\beta_2 \epsilon_2)^{\frac{1}{2}} + \overline{D}_{12} (\gamma_2 \epsilon_2)^{\frac{1}{2}} \cos \delta_2\right) \sin \varphi \quad \cos \psi_2 \\
&+ \overline{D}_{12} (\gamma_2 \epsilon_2)^{\frac{1}{2}} \sin \delta_2 \sin \varphi \quad \sin \psi_2
\end{aligned}$$

The maximum x from these three oscillating terms may be found by adding the coefficients of $\cos\psi_2$ and $\sin\psi_2$ quadratically, and then adding this result to the coefficient of $\cos\psi$. This gives

$$\begin{aligned} x_{\max} &= (\beta_1\epsilon_1)^{\frac{1}{2}}\cos\varphi + (\beta_{x,2}\epsilon_2)^{\frac{1}{2}}\sin\varphi \\ \beta_{x,2} &= \overline{D}_{11}^2\beta_2 + \overline{D}_{12}^2\gamma_2 + 2\overline{D}_{11}\overline{D}_{12}\alpha_2 \end{aligned} \tag{2.6a}$$

where u was made of the relationship

$$\cos\delta_2 = \alpha_2/\left(\beta_2\gamma_2\right)^{\frac{1}{2}}.$$

Equation (2.6a) shows that $x_{\max}(s)$ may be increased by the presence of linear coupling for reasons that include the following

1. the coupling may increase the beta functions β_1, β_2;
2. the coupling may cause an emittance mismatch. A particle which has the emittances ϵ_x, ϵ_y in the absence of coupling may have emittances ϵ_1, ϵ_2 in the presence of coupling which are larger than ϵ_x, ϵ_y.

In a similar way, one can find $y_{\max}$ using

$$y = u\cos\varphi - (D_{11}\ v + D_{12}p_v)\sin\varphi$$

and one finds

$$\begin{aligned} y_{\max} &= (\beta_2\epsilon_2)^{\frac{1}{2}}\cos\varphi + (\beta_{y,1},\epsilon_1)^{\frac{1}{2}}\sin\varphi \\ \beta_{y,1} &= D_{11}^2\beta_1 + D_{12}^2\gamma_1 + 2D_{11}D_{12}\alpha_1 \end{aligned} \tag{2.6b}$$

One can also find expressions for $x'_{\max}$ and $y'_{\max}$. Using

$$\begin{aligned} x' &= p_v\cos\varphi + \left(\overline{D}_{21}u + \overline{D}_{22}p_u\right)\sin\varphi \\ &= (\gamma_1\epsilon_1)^{\frac{1}{2}}\cos\left(\psi_1 - \delta_1\right) \\ &+ \overline{D}_{21}\sin\varphi\,(\beta_2\epsilon_2)^{\frac{1}{2}}\cos\psi_2 \\ &+ \overline{D}_{22}\sin\varphi\,(\gamma_2\epsilon_2)^{\frac{1}{2}}\cos\left(\psi_2 - \delta_2\right). \end{aligned}$$

one finds

$$\begin{aligned} x'_{\max} &= (\gamma_1\epsilon_1)^{\frac{1}{2}}\cos\varphi + (\gamma_{x,2}\epsilon_2)^{\frac{1}{2}}\sin\varphi \\ \gamma_{x,2} &= \overline{D}_{21}^2\beta_2 + \overline{D}_{22}^2\gamma_2 + 2\overline{D}_{21}\overline{D}_{22}\alpha_2 \end{aligned} \tag{2.6c}$$

Using

$$y' = p_u\cos\varphi - (D_{21}v + D_{22}p_v)\sin\varphi,$$

one finds

$$y'_{\max} = (\gamma_2\epsilon_2)^{\frac{1}{2}}\cos\varphi + (\gamma_{y,1}\epsilon_1)^{\frac{1}{2}}\sin\varphi$$
$$\gamma_{y,1} = D_{21}^2\beta_1 + D_{22}^2\gamma_1 + 2D_{21}D_{22}\alpha_1 \tag{2.6d}$$

3. $x_{\max}$ and $y_{\max}$ for a given ϵ_t

Consider all the particles that lie on the 4-dimensional surface in x, x', y, y' given by $\epsilon_t = \epsilon_1 + \epsilon_2 =$ constant. The question answered here is what is the largest x and the largest y reached by all particle lying on the surface $\epsilon_t =$ constant. Note that ϵ_t is the total emittance in the presence of linear coupling and ϵ_t is an invariant of the motion.

Each particle lying on the surface of constant ϵ_t will have emittances ϵ_1, ϵ_2 and its $x_{\max}$ and $y_{\max}$ may be computed from Eq. (2.6)

$$x_{\max} = \sqrt{\beta_1\epsilon_1}\cos\varphi + \sqrt{\beta_{x,2}\epsilon_2}\sin\varphi \tag{3.1a}$$

$$y_{\max} = \sqrt{\beta_2\epsilon_2}\cos\varphi + \sqrt{\beta_{y,1}\epsilon_1}\sin\varphi \tag{3.1b}$$

On the surface of constant ϵ_t, $\epsilon_1+\epsilon_2 = \epsilon_t$ and there is a choice of ϵ_1, ϵ_2 which maximizes $x_{\max}$ and another choice that maximizes $y_{\max}$. For example, for $x_{\max}$ one can replace ϵ_2 with $\epsilon_2 = \epsilon_t - \epsilon_1$ in Eq. (3.1) and find the maximum x, $\hat{x}_{\max}(s)$

$$\hat{x}_{\max} = \left(\hat{\beta}_x\epsilon_t\right)^{\frac{1}{2}}$$
$$\hat{\beta}_x = \left(\beta_1\cos^2\varphi + \beta_{x,2}\sin^2\varphi\right)^{\frac{1}{2}} \tag{3.2a}$$
$$\beta_{x,2} = \overline{D}_{11}^2\beta_2 + \overline{D}_{12}^2\gamma_2 + 2\overline{D}_{11}\overline{D}_{12}\alpha_2$$

For $\hat{y}_{\max}$, one finds

$$\hat{y}_{\max} = \left(\hat{\beta}_y\epsilon_t\right)^{\frac{1}{2}}$$
$$\hat{\beta}_y = \left(\beta_2\cos^2\varphi + \beta_{y,1}\sin^2\varphi\right)^{\frac{1}{2}} \tag{3.2b}$$
$$\beta_{y,1} = D_{11}^2\beta_1 + D_{12}^2\gamma_1 + 2D_{11}D_{12}\alpha_1$$

4. Examples of Using $x_{\max}$, $y_{\max}$ Results

4.1 Dynamic Aperture and Linear Coupling

Often the dynamic aperture is computed by finding the largest initial x, A_{SL}, which is stable for a specified number of turns and with the initial conditions x_0, y_0, $x'_o = y'_0 = 0$ and $\epsilon_{x_0} = \epsilon_{y_0}$. The tracking run is often done starting at a QF in a normal cell. It has been found[2] in RHIC that in the presence of linear coupling, A_{SL} can depend strongly on which QF the tracking run is started at. This can be understood, and estimated, by computing $x_{\max}$, using Eq. (2.6) at the high-β quadrupoles in the insertion region. This $x_{\max}$ depends on which QF the particle is started at, and thus explains the dependence of A_{SL} on the choice of QF at which the particle is started.

4.2 Aperture Requirements at Injection

The presence of linear coupling may increase the aperture requirements at injection. Let us assume that the injected beam in the absence of linear coupling, has maximum emittances ϵ_x and ϵ_y. The particles when injected in the presence of linear coupling may have a maximum total emittance ϵ_t such that ϵ_t is larger than $\epsilon_x + \epsilon_y$. This emittance mismatch is one source of the possible increase in aperture requirements. Another source is the increase in the beta functions due to linear coupling. The possible increase in aperture due to this source may be computed using Eq. (3.2a) $\hat{x}_{\max} = \left(\hat{\beta}_x \epsilon_t\right)^{1/2}$.

5. Effect of Solenoidal Fields

The previous results were obtained assuming no solenoidal fields were present. If solenoidal fields are present, the expressions for $x_{\max}$, $y_{\max}$ are unchanged. However φ and D which appear in the expressions for $x_{\max}$ and $y_{\max}$, will be changed by the presence of solenoids. The results for $x'_{\max}$ and $y'_{\max}$ are unchanged outside the solenoids.

If solenoids are present, then $p_x = x'$ and $p_y = y'$ are replaced by

$$\begin{aligned} p_x &= x' + Ly \\ p_y &= y' - Lx, \end{aligned} \tag{5.1}$$

where $L = B_s/2B\rho$, and B_s is the longitudinal field in the solenoidal. The results for $x'_{\max}$ and $y'_{\max}$ are still valid if $x'_{\max}$ is replaced by $p_{x,\max}$, and $y'_{\max}$ by $p_{y,\max}$. This shows that outside the solenoids, when $L = 0$, the results for $x'_{\max}$ and $y'_{\max}$ are unchanged.

References

1. D. Edwards and L. Teng, IEEE 1973 PAC, p. 885 (1973).
2. G. Parzen, BNL AD/RHIC-94 (1991); and Proc. IEEE 1991 PAC (to be published).