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Beta Functions, the Normal Mode Rotation Angle and Eigenfunctions in the Presence of Linear Coupling

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1. Introduction

This paper finds expressions for the beta functions and the normal mode rotation angle in the presence of linear coupling. To do this, expressions will be found for the 4 eigenfunctions of the transfer matrix using perturbation theory techniques. It will be shown that having these results for the 4 eigenfunction is equivalent to having a result for the transfer matrix, and results for the beta function and the normal mode rotation angle can then be found from these results for the eigenfunctions.

These results for the beta function and the rotation angle parameter suggest ways of controlling these effects by controlling certain important driving terms. The same driving terms appear in the expressions for the beta functions and for the normal mode rotation angle parameter, and also in the previously found results[1] for the tune shift including the higher order tune shift. For all these three effects, the harmonics of the skew quadrupole field close to $\nu_x + \nu_y$ are found to be the important harmonics.

The problem of finding expressions for the beta function has been treated in a previous paper.[2] However, the methods used in this paper are an improvement over the previous method, and allows other problems to be treated such as the normal mode rotation angle, emittance blow up at injection and finding expressions for the other orbit parameters that occur in linear coupling. The eigenfunction results given in this paper will be applied to the problem of emittance distortion at injection due to linear coupling in a future paper. Expressions for the emittance distortions will be found which contain the same driving

terms as those that appear in the results for the beta function distortion, the normal mode rotation angle parameter and the higher order tune shift. It appears possible to globally correct all these effects with a skew quadrupole correction system that controls the harmonics of the skew quadrupole field that are close to $\nu_x + \nu_y$.

2. The Eigenfunctions

This paper will find perturbation theory expressions for the 4 eigenfunctions of the 4×4 transfer matrix in the presence of linear coupling. These expressions for the eigenfunctions can be useful in studying various effects due to linear coupling, such as the distortion of the beta function, the distortion in the emittance, and the rotation of the plane of the normal modes. This paper will find expressions for the eigenfunctions which are valid when the tune is close to a linear coupling resonance and to first order in the skew quadrupole field generating the linear coupling. These results will be applied to the problems of the beta function distortion, and to the normal mode rotation angle.

The eigenfunctions may be defined in terms of the transfer matrix, $T(s, s_0)$,

$$x(s) = T(s, s_0)\ x(s_0) \tag{2.1a}$$

In Eq. (2.1a), $T(s, s_0)$ is a 4×4 matrix, $x(s)$ is a 4×1 column vector

$$x = \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} \tag{2.1b}$$

In the absence of solenoids, $p_x = x'$ and $p_y = y'$. The eigenfunctions are those $x(s)$ that satisfy

$$T(s + L, s)\, x = \lambda x, \tag{2.1c}$$

where L is the period of the magnetic guide field.

It can be shown,[3] that there are 4 eigenfunctions $x_i(s)$, $i = 1, 2, 3, 4$ with eigenvalues λ_i, and which occur in pairs such that for stable motion,

$$x_2 = x_1^*, \quad x_4 = x_3^*, \quad \lambda_2 = \lambda_1^*, \quad \lambda_4 = \lambda_3^*.$$

It can be shown[4] that the eigenfunctions are solutions of the equations of motions, and that

$$\begin{aligned} x_1(s) &= \exp(i2\pi\nu_1 s/L) \quad f_1(s), \\ x_3(s) &= \exp(i2\pi\nu_2 s/L) \quad f_2(s). \end{aligned} \tag{2.2}$$

$f_1(s)$, $f_2(s)$ are periodic in s with period L, and ν_1, ν_2 are the normal mode tunes. Note x_i and f_i are both 4×1 column vectors.

The equations of motion can be written as[1]

$$\begin{aligned} \left(\frac{d^2}{d\theta_x^2} + \nu_x^2\right)\eta_x &= b_x(s)\,\eta_y \\ \left(\frac{d^2}{d\theta_y} + \nu_y^2\right)\eta_y &= b_y(s)\,\eta_x \end{aligned} \tag{2.3}$$

$$\begin{aligned} x &= \beta_x^{1/2}\eta_x, \qquad y = \beta_y^{1/2}\eta_y \\ \theta_x &= \int ds\,(1/\nu_x\beta_x), \qquad \theta_y = \int ds\,(1/\nu_y\beta_y) \\ b_x(s) &= \nu_x^2\beta_x(\beta_x\beta_y)^{1/2}\,a_1/\rho \\ b_y(s) &= \nu_y^2\beta_y(\beta_x\beta_y)^{1/2}\,a_1/\rho \end{aligned}$$

The skew quadrupole field is described by $a_1(s)$. On the median plane, the field B_x is given by

$$B_x = -B_0\ a_1\ x,$$

where B_0 is the main dipole field. ρ is the radius of curvature of the main dipole.

The solutions of Eq. (2.3) which have the form of Eq. (2.2) were found in two previous papers[1,2] when ν_x, ν_y are near the resonance line $\nu_x = \nu_y + p$. These solutions may be written as

$$\begin{aligned} \eta_x &= A\exp(i\nu_{x,s}\theta_x)\left\{1 + \sum_{n\neq -p} f_n\right\} \\ f_n &= \frac{(\nu_{xs} - \nu_x)}{\Delta\nu}\frac{2\nu_x b_n \exp[-i(n+p)\theta_x]}{(n - \nu_x - \nu_y)(n+p)} \end{aligned} \tag{2.4}$$

$$\eta_y = B\exp\left(i\nu_{ys},\theta_y\right)\left\{1+\sum_{n=p} g_n\right\}$$

$$g_n = \frac{\left(\nu_{ys}-\nu_y\right)}{\Delta\nu^*}\frac{2\nu_y c_n \exp\left[-i\left(n-p\right)\theta_y\right]}{\left(n-\nu_x-\nu_y\right)\left(n-p\right)}$$

$$\Delta\nu = \left(1/4\pi\rho\right)\int ds\left(\beta_x\beta_y\right)^{\frac{1}{2}} a_1 \exp\left[i\left(-\nu_{xs}\theta_x+\nu_{ys}\theta_y\right)\right]$$

$$b_n = \frac{1}{4\pi\rho}\int ds\left(\beta_x\beta_y\right)^{\frac{1}{2}} a_1 \exp\left[i\left(\left(n-\nu_y\right)\theta_x+\nu_y\theta_y\right)\right]$$

$$c_n = \frac{1}{4\pi\rho}\int ds\left(\beta_x\beta_y\right)^{\frac{1}{2}} a_1 \exp\left[i\left(\nu_x\theta_x+\left(n-\nu_x\right)\theta_y\right)\right]$$

$$\theta_x = \psi_x/\nu_x, \qquad \theta_y = \psi_y/\nu_y$$

ν_{xs} and ν_{ys} are the solutions of

$$\nu_{xs} = \nu_{ys}+p, \qquad \left(\nu_{xs}-\nu_x\right)\left(\nu_{ys}-\nu_y\right) = \left|\Delta\nu\right|^2 \tag{2.5}$$

There are two solutions of Eq. (2.5) corresponding to the two normal modes. For the mode for which $\nu_{xs}\to\nu_x$ when $a_1\to 0$, we will put $\nu_{xs}=\nu_1$, $\nu_{ys}=\nu_1-p$. For the mode for which $\nu_{ys}\to\nu_y$ when $a_1\to 0$, we will put $\nu_{ys}=\nu_2$, $\nu_{xs}=\nu_2+p$. The A and B coefficients are related by

$$\begin{aligned} B_1 &= \frac{-\left(\nu_1-\nu_x\right)}{\Delta\nu}A_1 \quad \text{for the } \nu_1 \text{ mode} \\ A_2 &= \frac{-\left(\nu_2-\nu_y\right)}{\Delta\nu^*}B_2 \quad \text{for the } \nu_2 \text{ mode} \end{aligned} \tag{2.6}$$

The results for the eigenfunctions, Eq. (2.3) were found by solving the equations of motion to first order terms in a_1. It has been assumed that ν_x,ν_y the unperturbed tune, is close to the coupling resonance $\nu_x=\nu_y+p$ and the $\nu_x-\nu_y-p$ can be considered to be small, of the same order as a_1. This last assumption allows the equations to be simplified and it is the case of most interest to us.

The A and B coefficients in Eq. (2.5) have now to be chosen so that the eigenfunctions are properly normalized, which means the eigenfunctions can be then expressed in terms of the orbit parameters like β_1,α_1,ψ_1 and β_2,α_2,ψ_2. To understand this better consider the 2 dimensional case. If we wish the eigenfunction to be related to β,ψ by

$$x = \beta^{1/2}\exp\left(i\psi\right),$$

then

$$p_x = x' = \beta^{-1/2}\left(-\alpha+i\right)\exp\left(i\psi\right)$$

and the two eigenfunctions are given by x_1, x_1^* where

$$x_1 = \begin{bmatrix} \beta^{\frac{1}{2}} \\ \beta^{-\frac{1}{2}}(-\alpha + i) \end{bmatrix} \exp(i\psi) \tag{2.7}$$

These eigenfunctions are normalized so that

$$\tilde{x}_1^* \; S \; x_1 = 2i \tag{2.8}$$

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$\tilde{x}_1$ is the transpose of x_1.

The normalization given by Eq. (2.8) gives the relationship between x_1 and β, α, ψ given by Eq. (2.7). It is shown in section 6, that in the 4 dimensional case the normalization Eq. (2.8) will allow the eigenfunctions x_1, x_3 to be related to $\beta_1\alpha_1\psi_1$ and $\beta_2\alpha_2\psi_2$ in a corresponding way. In this case, S is now the 4×4 matrix

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{2.9}$$

Eq. (2.8) will be used to determine the coefficients A, B. This gives the relationship, see section 5,

$$|A|^2 (\nu_{xs}/\nu_x) + |B|^2 (\nu_{ys}/\nu_y) = 1. \tag{2.10}$$

Eq. (2.10) together with Eq. (2.6) determine A and B.

For the ν_1 mode

$$B_1 = -\frac{\nu_1 - \nu_x}{\Delta\nu} A_1,$$

$$|A_1|^2 \left(\frac{\nu_1}{\nu_x} + \frac{\nu_1 - p}{\nu_y} \left| \frac{\nu_1 - \nu_x}{\Delta\nu} \right|^2 \right) = 1. \tag{2.11a}$$

For the ν_2 mode

$$A_2 = -\frac{\nu_2 - \nu_y}{\Delta\nu^*} B_2,$$

$$|B_2|^2 \left(\frac{\nu_2}{\nu_y} + \frac{\nu_2 + p}{\nu_x} \left| \frac{\nu_2 - \nu_y}{\Delta\nu} \right|^2 \right) = 1. \tag{2.11b}$$

A case of particular interest is when the linear coupling has been corrected to make $\Delta\nu \simeq 0$. There are then two solutions of interest,

1. $|\Delta\nu| \ll |\nu_x - \nu_y - p|$
2. $|\nu_x - \nu_y - p| \ll |\Delta\nu|$

In case 1., $\Delta\nu$ has been made small enough so that the tune ν_x, ν_y is well outside the width of the difference resonance. This may not always be achieved. ν_x, ν_y may be very close to the difference resonance, and the best setting of the correction system to minimize the tune splitting does not have to correspond to $\Delta\nu = 0$.

If $|\Delta\nu| \ll |\nu_x - \nu_y - p|$, one finds (see section 5)

$$|A_1| = 1, \qquad B_1 = 0,$$

$$|B_2| = 1, \qquad A_2 = 0.$$

The two modes appear to be decoupled.

If $|\nu_x - \nu_y - p| \ll |\Delta\nu|$, one finds

$$|A_1| = |B_1| = 1/\sqrt{2}$$

$$|A_2| = |B_2| = 1/\sqrt{2}$$

The two modes appear completely coupled.

3. The Transfer Matrix in Terms of the Eigenfunctions

Given the four eigenfunctions $x_i, i = 1, 4$ which are normalized such that

$$\tilde{x}_i^* \, S x_i = 2i, \tag{3.1}$$

then it will be shown that one can find the transfer matrix $T(s, s_0)$ from

$$\begin{aligned} T(s, s_0) &= (-1/2i)\, U(s)\, \overline{U}(s_0) \\ \overline{U} &= \tilde{S}\, \tilde{U} S \\ U &= [x_1 \; x_2 \; x_3 \; x_4]. \end{aligned} \tag{3.2}$$

U is a 4×4 matrix and x_i is a 4×1 column vector.

Eq. (4.2) will be derived for the 2-dimensional case. The generalization to 4 or more dimensions is clear. In two dimensions a solution of the equation of motion can be written as

$$\begin{aligned} x &= a_1 x_1 + a_2 x_2, \quad x_2 = x^*, \\ a_1 &= \tilde{x}_1^* \, S x / 2i, \; a_2 = a_1^* = \tilde{x}_2^* \, S x / (-2i) \end{aligned} \tag{3.3}$$

Evaluate a_1 and a_2 using $x(s_0)$. Then

$$\begin{aligned} x &= (1/2i)\left(x_1(s)\,\tilde{x}_1^*(s_0) - x_2(s)\,\tilde{x}_2^*(s_0)\right) S\, x(s_0) \\ x &= (1/2i)\left(x_1(s)\,\tilde{x}_1^*(s_0) - x_1^*(s)\,\tilde{x}_1(s_0)\right) S\, x(s_0) \\ x &= (1/2i)\left[x_1(s) \quad x_1^*(s)\right] S \begin{bmatrix} \tilde{x}_1(s_0) \\ \tilde{x}_1^*(s_0) \end{bmatrix} S\, x(s_0) \\ x &= (-1/2i)\, U(s)\, \overline{U}(s_0)\; x(s_0). \end{aligned} \tag{3.4}$$

Thus

$$T(s, s_0) = (-1/2i)\, U(s)\, \overline{U}(s_0) \tag{3.5}$$

One may note that $V = (-2i)^{-\frac{1}{2}}\; U(s)$ is symplectic as $T(s,s) = I$ and $V\overline{V} = I$.

Eq. (3.5) shows that knowing the eigenfunctions x_i is equivalent to knowing the transfer matrix $T(s, s_0)$. Eq. (3.5) also shows that $T(s, s_0)$ is symplectic as it is the product of two symplectic matrices, $V(s)$ and $\overline{V}(s_0)$.

4. Applications of the Eigenfunction Results

The results found for the eigenfunctions in section 2 will be used to compute the orbit parameters β_1 and β_2, the beta functions of the normal modes, and φ, the normal mode rotation angle. To compute the orbit parameters one needs the relationship between the eigenfunctions and the orbit parameters.

In 2 dimensions there are just 3 orbit parameters and the eigenfunctions are related to these 3 parameters by

$$x_1 = \begin{bmatrix} \beta^{1/2} \exp(i\psi) \\ \beta^{-1/2}(-\alpha + i) \exp(i\psi) \end{bmatrix} \tag{4.1}$$

In 4 dimensions, there are 10 orbit parameters. These include $\beta_1, \alpha_1, \psi_1$ and $\beta_2, \alpha_2, \psi_2$ of the two normal modes. In addition there are 4 parameters that define the transformation to the normal mode coordinates.[7] These 4 parameters may be denoted by φ and D where D is a 2×2 matrix and $|D| = I$. In terms of these 10 orbit parameters the eigenfunctions are given by, see section 6,

$$\begin{aligned} x_1 &= \begin{bmatrix} \cos\varphi\; \mu_1 \\ -D \sin\varphi\; \mu_1 \end{bmatrix} \qquad x_3 = \begin{bmatrix} \overline{D} \sin\varphi\; \mu_2 \\ \cos\varphi\; \mu_2 \end{bmatrix} \\ \mu_1 &= \begin{bmatrix} \beta_1^{\frac{1}{2}} \\ \beta_1^{-\frac{1}{2}}(-\alpha_1 + i) \end{bmatrix} \exp(i\psi_1) \quad \mu_2 = \begin{bmatrix} \beta_2^{\frac{1}{2}} \\ -\beta_2^{-\frac{1}{2}}(-\alpha_2 + i) \end{bmatrix} \exp(i\psi_2) \end{aligned} \tag{4.2}$$

where $\overline{D} = D^{-1}$.

If the eigenfunctions are known, then Eqs. (4.2) can be inverted to find the 10 orbit parameters. One can use additional relationships.

$$\frac{d\psi_1}{ds} = \frac{1}{\beta_1}, \quad \frac{d\psi_2}{ds} = \frac{1}{\beta_2} \tag{4.3}$$

$$\alpha_1 = -\frac{1}{2}\beta_1' + \beta_1 \tan\varphi d\varphi/ds, \quad \alpha_2 = -\frac{1}{2}\beta_2' + \beta_2 \tan\varphi d\varphi/ds$$

which are valid in absence of solenoidal fields. Eqs. (4.1) and the results for the eigenfunctions (Eqs. (2.3) will now be used to find β_1 and β_2.

For the ν_1 mode, using the first element of x_1 one finds

$$\cos\varphi\beta_1^{\frac{1}{2}} \exp(i\psi_1) = x_{11}$$

$$\begin{aligned} x_{11} &\equiv A_1\beta_x^{\frac{1}{2}} \exp(i\nu_1\theta_x)\left(1 + \sum_{n\neq -p} f_n\right) \\ &= A_1\beta_x^{\frac{1}{2}}\left(1 + \frac{1}{2}\sum_{n\neq -p}(f_n + f_n^*)\right)\exp\left[i\left(\nu_1\theta_x + \frac{1}{2i}\sum_{n\neq -p}(f_n - f_n^*)\right)\right]. \end{aligned} \tag{4.4}$$

Thus we find for ψ_1

$$\psi_1 = \nu_1\theta_x + \frac{1}{2i}\sum_{n\neq -p}(f_n - f_n^*). \tag{4.5a}$$

From ψ_1 one can find β_1 from $1/\beta_1 = d\psi_1/ds$

$$\frac{1}{\beta_1} = \frac{\nu_1}{\nu_x\beta_x} + \frac{1}{2\nu_x\beta_x}\sum_{n\neq -p}(-n-p)(f_n + f_n^*), \tag{4.5b}$$

$$\frac{\beta_1 - \beta_x}{\beta_x} = -\sum_{\text{all } n}\left\{\frac{\nu_1 - \nu_x}{\Delta\nu}\frac{b_n}{n - \nu_x - \nu_y}\exp\left[-i(n+p)\theta_x\right] + \text{c.c.}\right\}. \tag{4.6a}$$

Using the third element of x_3, one finds for β_2

$$\frac{\beta_2 - \beta_y}{\beta_y} = -\sum_{\text{all } n}\left\{\frac{\nu_2 - \nu_y}{\Delta\nu^*}\frac{c_n}{(n - \nu_x - \nu_y)}\exp\left[-i(n-p)\theta_y\right] + \text{c.c.}\right\}. \tag{4.6b}$$

Eqs. (4.6) show that the dominant driving terms b_n, c_n are those for which $n \simeq \nu_x + \nu_y$. Close to the coupling resonance, when $|\nu_x - \nu_y - p| \ll |\Delta\nu|$, then $(\beta_1 - \beta_x)/\beta_x$ is linear in a_1 since $|(\nu_1 - \nu_x)/\Delta\nu| \simeq 1$ when ν_x, ν_y are close to the coupling resonance. Far from the coupling resonance, when $|\nu_x - \nu_y - p| \gg |\Delta\nu|$, then $|\nu_1 - \nu_x|/|\Delta\nu| \simeq |\Delta\nu|$ and $(\beta_1 - \beta_x)/\beta_x$ is quadratic in a_1.

4.1 Normal Mode Rotation Angle, φ

Using Eq. (4.4) for the ν_1 mode one finds

$$\cos\varphi\beta_1^{\frac{1}{2}} = |A_1|\beta_x^{\frac{1}{2}}\left(1+\frac{1}{2}\sum_{n\neq -p}(f_n+f_n^*)\right) \tag{4.7}$$

and

$$\cos\varphi = |A_1|\,(\beta_x/\beta_1)^{\frac{1}{2}}\left(1+\frac{1}{2}\sum_{n\neq -p}(f_n+f_n^*)\right).$$

From (4.5b)

$$\frac{\beta_x}{\beta_1} = 1+\frac{\nu_1-\nu_x}{\nu_x}-\frac{1}{2\nu_x}\sum_{n\neq -p}(n+p)\,(f_n+f_n^*) \tag{4.8}$$

$$\cos\varphi = |A_1|\left(1+\frac{1}{2}\frac{\nu_1-\nu_x}{\nu_x}+\sum_{n\neq -p}(f_n+f_n^*)\left(\frac{1}{2}-\frac{n+p}{4\nu_x}\right)\right) \tag{4.9a}$$

$$f_n = \frac{\nu_1-\nu_x}{\Delta\nu}\frac{2\nu_x\, b_n}{(n-\nu_x-\nu_y)\,(n+p)}\exp\left[-i\,(n+p)\,\theta_x\right]$$

$$|A_1|^2\left(\frac{\nu_1}{\nu_x}+\frac{\nu_1-p}{\nu_y}\left|\frac{\nu_1-\nu_x}{\Delta\nu}\right|^2\right) = 1$$

One can find another expression for $\cos\varphi$ by using the x_3 eigenfunction. This gives

$$\cos\varphi = |B_2|\,(\beta_y/\beta_2)^{\frac{1}{2}}\left(1+\frac{1}{2}\sum_{n\neq p}(g_n+g_n^*)\right)$$

$$\cos\varphi = |B_2|\left(1+\frac{1}{2}\frac{\nu_2-\nu_y}{\nu_y}+\sum_{n\neq p}(g_n+g_n^*)\left(\frac{1}{2}-\frac{n-p}{4\nu_y}\right)\right) \tag{4.9b}$$

$$g_n = \frac{\nu_2-\nu_y}{\Delta\nu^*}\frac{2\nu_y c_n}{(n-\nu_x-\nu_y)\,(n-p)}\exp\left[-i\,(n-p)\,\theta_y\right]$$

$$|B_2|^2\left(\frac{\nu_2}{\nu_y}+\frac{\nu_2+p}{\nu_x}\left|\frac{\nu_2-\nu_y}{\Delta\nu}\right|^2\right) = 1.$$

The results for the beta functions, Eq. (4.6) and the results for $\cos\varphi$, Eq. (4.9), show that they have same important driving terms b_n, c_n for $n \simeq \nu_x+\nu_y$. The higher order ν-shift also has the same driving terms.[1] Thus a correction system that corrects these driving terms might be able to correct all these three effects simultaneously. In addition

to the driving terms b_n, c_n, $n \simeq \nu_x + \nu_y$, a complete linear coupling corrections would also have to correct $b_{-p} \simeq c_p^*$ which drives the nearby difference resonance and the lowest order tune shift.

5. Calculation of the A and B Coefficients

In section 2, expressions for the eigenfunctions were given in Eq. (2.3) which contained two normalization coefficients A and B and it was stated that A and B are determined by the normalization conditions

$$\begin{aligned} \tilde{x}_1^* \, S \, x_1 &= 2i \\ \tilde{x}_3^* \, S \, x_3 &= 2i \end{aligned} \tag{5.1}$$

The relationship (Eq. 2.8) between A and B will be found in this section using the conditions Eq. (5.1).

It is convenient to go from the x, p_x, y, p_y variables to $\eta_x, p_{\eta x}, \eta_y, p_{\eta y}$ variables,

$$\begin{aligned} \begin{pmatrix} x \\ p_x \end{pmatrix} &= G_x \begin{pmatrix} \eta_x \\ p_{\eta x} \end{pmatrix} \\ G_x &= \begin{pmatrix} \sqrt{\beta_x} & 0 \\ -\frac{\alpha}{\sqrt{\beta_x}} & \frac{1}{\sqrt{\beta_x}} \end{pmatrix} \end{aligned} \tag{5.2}$$

G_x is a symplectic, $|G_x| = 1$. In a similar way y, p_y are related to $\eta_y, p_{\eta y}$ by G_y, and in 4-dimensions

$$\begin{aligned} x &= G \, \eta \\ G &= \begin{bmatrix} G_x & 0 \\ 0 & G_y \end{bmatrix} \\ x &= \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_x \\ p_{\eta x} \\ \eta_y \\ p_{\eta y} \end{pmatrix} \end{aligned} \tag{5.3}$$

Because G is symplectic, Eq. (5.1) becomes

$$\tilde{x}^* \, S \, x = \tilde{\eta}^* S \, \eta = 2i \tag{5.4}$$

one then finds

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{5.5}$$

$$\eta^* S\eta = (\eta_x^* p_{\eta x} - cc) + (\eta_y^* p_{\eta y} - cc) = 2i$$

$$\mathrm{Im}\,(\eta_x^* p_{\eta x}) + \mathrm{Im}\,(\eta_y^* p_{\eta y}) = 1 \tag{5.6}$$

From Eq. (2.3)

$$\eta_x = A \exp(i\nu_{xs}\theta_x)\left(1 + \sum_{n\neq -p} f_n\right), \tag{5.7a}$$

and it can be shown that $p_{\eta x} = (1/\nu_x)\, d\eta_x/d\theta_x$ so

$$p_{\eta x} = A\ \exp(i\nu_{xs}\theta_x)\, i\left[\frac{\nu_{xs}}{\nu_x}\left(1 + \sum_{n\neq -p} f_n\right) + \frac{1}{\nu_x}\sum_{n\neq -p}(-(n+p))\, f_n\right] \tag{5.7b}$$

$$\eta_x^* p_{etax} = |A|^2 i\left[\frac{\nu_{xs}}{\nu_x} + \sum_{n\neq -p}(f_n^* + f_n)\frac{\nu_{xs}}{\nu_x} + \frac{1}{\nu_x}\sum_{n\neq -p}(-(n+p))\, f_n\right] \tag{5.8a}$$

In some way one finds

$$\eta_y^* p_{\eta y} = |B|^2 i\left\{\frac{\nu_{ys}}{\nu_y} + \sum_{n\neq p}(g_n^* + g_n)\frac{\nu_{ys}}{\nu_y} + \frac{1}{\nu_y}\sum_{n\neq p}(-(n-p))\, g_n\right\} \tag{5.8b}$$

Since $\mathrm{Im}\,(\eta_x^* p_{\eta x} + \eta_y^* p_{\eta y}) = 1$, when Eqs. (5.8a) and (5.8b) are added together, the terms depending on θ_x and θ_y have to cancel each other. Only the constant terms remain. Eq. (5.6) then gives

$$|A|^2\nu_{xs}/\nu_x + |B|^2\nu_{ys}/\nu_y = 1 \tag{5.9a}$$

Eq. (5.9) together with Eq. (2.4)

$$\begin{aligned} B_1 &= \frac{-(\nu_1 - \nu_x)}{\Delta\nu} A_1 \\ A_2 &= \frac{-(\nu_2 - \nu_y)}{\Delta\nu^*} B_2 \end{aligned} \tag{5.9b}$$

determine A and B for the two modes.

The solutions of Eq. (5.9) for A and B are particularly interesting when the difference resonance has been corrected so that $\Delta\nu \simeq 0$. There are then two solutions of interest

$$|\Delta\nu| \ll |\nu_x - \nu_y - p|$$

$$|\nu_x - \nu_y - p| \ll |\Delta\nu|$$

For the sake of simplicity, the solutions will be found for case when $p = 0$, and ν_x, ν_y is near the $\nu_y - \nu_y = 0$ resonance.

For the case when $|\Delta\nu| \ll |\nu_x - \nu_y|$ then, from Eq. (2.5),

$$\begin{aligned} \nu_1 &= \nu_x \pm 2|\Delta\nu|^2/|\nu_x - \nu_y| \\ \nu_2 &= \nu_y \mp 2|\Delta\nu|^2/|\nu_x - \nu_y| \\ |B_1| &= \left|\frac{2\Delta\nu}{\nu_x - \nu_y}\right| |A_1| \ll |A_1|, \\ |A_2| &= \left|\frac{2\Delta\nu}{\nu_x - \nu_y}\right| |B_2| \ll |B_2|. \end{aligned} \tag{5.10}$$

For the ν_1 mode

$$\begin{aligned} |A_1| &\simeq 1 \\ |B_1| &\simeq 0 \end{aligned} \tag{5.11a}$$

for the ν_2 mode

$$\begin{aligned} |A_2| &\simeq 0 \\ |B_2| &\simeq 1 \end{aligned} \tag{5.11b}$$

For the case when $|\Delta\nu| \gg |\nu_x - \nu_y|$

$$\begin{aligned} \nu_1 &= \frac{1}{2}(\nu_x + \nu_y) \pm |\Delta\nu| \\ \nu_2 &= \frac{1}{2}(\nu_x + \nu_y) \mp |\Delta\nu| \\ B_1 &= -\frac{(\nu_x + \nu_y)/2 \pm |\Delta\nu| - \nu_x}{\Delta\nu} A_1 \\ B_1 &= \pm(|\Delta\nu|/\Delta\nu) A_1 \\ A_2 &= \mp(|\Delta\nu|/\Delta\nu^*) B_2 \end{aligned} \tag{5.12a}$$

Thus one finds

$$|B_1| = |A_1| \quad \text{and} \quad |B_2| = |A_2|, \tag{5.12b}$$

The motion is completely coupled. Assuming that ν_x, ν_y are close to the resonance, Eq. (5.9a) gives

$$\begin{aligned} |A_1| &= |B_1| = 1/\sqrt{2} \\ |A_2| &= |B_2| = 1/\sqrt{2} \end{aligned} \tag{5.13}$$

From Eqs. (5.12) one also gets the relationship

$$A_1^* B_1 + A_2^* B_2 = 0 \tag{5.14}$$

6. The Eigenfunction and the 10 Orbit Parameter

In this section, the eigenfunctions will be related to the 10 orbit parameters for the coupled motion.

In two dimensions, the eigenfunction is related to the 3 orbit parameters β, α, ψ by

$$x_1 = \begin{pmatrix} \beta^{\frac{1}{2}} \\ \beta^{-\frac{1}{2}}(-\alpha + i) \end{pmatrix} \exp(i\psi) \tag{6.1}$$

and $x_2 = x_1^*$, x_1 obeys the normalization condition

$$\tilde{x}_1^* \; S \; x_1 = 2i \tag{6.2}$$

In four dimensions, one can go from the coordinates, x, p_x, y, p_y to an uncoupled set of coordinates v, p_v, u, p_u the normal coordinates, by the transformation[5]

$$x = R\, v \tag{6.3}$$

$$R = \begin{pmatrix} I\cos\varphi & \overline{D}\sin\varphi \\ -D\sin\varphi & I\cos\varphi \end{pmatrix} \tag{6.4}$$

I and R are 2×2 matrices. I is the 2×2 identity matrix. $\overline{D} = D^{-1}$ and $|D| = 1$. R is a symplectic matrix,

$$\begin{aligned} \overline{R}R &= I \\ \overline{R} &= \tilde{S}\tilde{R}S \end{aligned} \tag{6.5}$$

φ and the 3 independent elements of D may be considered as 4 of the orbit parameters. They are periodic in s. The other 6 orbit parameters are the $\beta_1, \alpha_1, \psi_1$ and $\beta_2, \alpha_2, \psi_2$ of the 2 normal modes.

It can be shown that $\tilde{x}^* \; Sx$ is a constant[3] of the motion. Also if x and v are related by a symplectic matrix then

$$\tilde{x}^* \; Sx = \tilde{v}^* Sv \tag{6.6}$$

The transfer matrix for v coordinates is given by

$$\begin{aligned} v(s) &= U(s, s_0)\, v(s_0) \\ U &= \overline{R}(s)\, T R(s_0) \end{aligned} \tag{6.7}$$

It can then be shown that the eigenfunction of U and v_i, and the eigenfunctions of T are related by

$$x_i = R\, v_i \tag{6.8}$$

The v coordinates are uncoupled, so the v_i eigenfunctions can be written down using Eq. (6.1) as

$$
\begin{gathered}
v_1 = \begin{pmatrix} \mu_1 \\ 0 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 0 \\ \mu_2 \end{pmatrix} \\
\mu_1 = \begin{pmatrix} \beta_1^{\frac{1}{2}} \\ \beta_1^{-\frac{1}{2}}(-\alpha_1 + i) \end{pmatrix} \exp(i\psi_1), \quad \mu_2 = \begin{pmatrix} \beta_2^{\frac{1}{2}} \\ \beta_2^{-\frac{1}{2}}(\alpha_2 + i) \end{pmatrix} \exp(i\psi_2) \\
v_2 = v_1^*, \qquad v_4 = v_3^*
\end{gathered}
\tag{6.9}
$$

one may note that $\tilde{v}_1^* S v_1 = \tilde{v}_3^* S v_3 = 2i$. The x_i can then be written down using $x = Rv$ as

$$
\begin{gathered}
x_1 = \begin{pmatrix} \mu_1 \cos\varphi \\ -D\mu_1 \sin\varphi \end{pmatrix} \\
x_3 = \begin{pmatrix} \overline{D}\mu_2 \sin\varphi \\ \mu_2 \cos\varphi \end{pmatrix}
\end{gathered}
\tag{6.10}
$$

Eq. (6.10) relates the eigenfunctions x_i to the 10 orbit parameters. Also $\tilde{x}_1^*, Sx_1 = \tilde{x}_3^* Sx_3 = 2i$.

7. References

1. G. Parzen, Theory of the Tune Shift Due to Linear Coupling, BNL Report AD/RHIC-100 (1991).
2. G. Parzen, Theory of the Beta Function Shift Due to Linear Coupling, BNL Report AD/RHIC-102 (1991).
3. E.D. Courant and H. Snyder, Theory of the AG Synchrotron, Annals of Physics, 3, No. 1, p. 1 (1958).
4. F. Willeke and G. Ripken, Methods of Beam Optics, DESY 88-114 (1988).
5. D. Edwards and L. Teng, Parameterization of Linear Coupled Systems, IEEE PAC p. 885 (1973).